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STRICT INCLUSION $O_{HB}<O_{HD}$ FOR ALL DIMENSIONS

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Let M be a Riemannian n-manifold ($n\geq 2$) and $HX(M)$ the real vector space of X-harmonic functions on M. Here X stands for B=bounded, D=Dirichlet-finite, or BD=bounded and Dirichlet-finite. A Riemannian manifold M belongs to the null class O_{HX}, by definition, if the class $HX(M)$ reduces to the real number field.

The purpose of the present paper is to show that the inclusion relation $O_{HB}\subset O_{HD}$ is strict for all $n\geq 3$. The result was announced in Kwon [3].

1. A real-valued C^2-function u on a Riemannian n-manifold M is said to be harmonic on M if it satisfies $\Delta u=0$, where

$$\Delta u=-\frac{1}{\sqrt{g}}\sum_{i,j=1}^{n}\frac{\partial}{\partial x^i}\left(\sqrt{g}\,g^{ij}\frac{\partial u}{\partial x^j}\right)$$

is the Laplace-Beltrami operator for M. Here (g_{ij}) is the Riemannian metric tensor for M, $(g^{ij})=(g_{ij})^{-1}$, and $g=\det(g_{ij})$. The Dirichlet integral of a C^1-function u on M is defined by

$$D(u)=\int_M \sum_{i,j=1}^{n} g^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j}dV$$

where $dV=\sqrt{g}\,dx^1\wedge dx^2\wedge\cdots\cdots\wedge dx^n$ is the volume element.

One of the important open problems in the classification theory is to determine whether or not the inclusion

$$O_{HB}\subset O_{HD}$$

is strict (cf. Sario-Nakai [5, p.406]). The problem was raised in 1950 from the Virtanen identity

$$O_{HD}=O_{HBD}(\supset O_{HB})$$

(Virtanen [71]). In 1953 Tôki [6] constructed a Riemann surface for the strictness when $n=2$, which bears his name. We shall follow a simpler construction by Sario [4].

In the present paper we shall construct a Riemannian n-manifold N ($n\geq 3$) such that dim $HB(N)=2$ and dim $HD(N)=1$. Thus the inclusion $O_{HB}\subset O_{HD}$ is strict for all $n\geq 2$.

2. Let $n\geq 3$. Denote by M_0 the punctured Euclidean n-space R^n-0 with the Riemannian structure

$$g_{ij}(x)=|x|^{-4}(1+|x|^{n-2})^{\frac{4}{n-2}}\delta_{ij},\quad 1\leq i,j\leq n,$$

where $|x|=[\sum_{i=1}^{n}(x^i)^2]^{\frac{1}{2}}$ for $x=(x^1, x^2, \cdots\cdots, x^n)\in M_0$.

First we observe

LEMMA 1. *A function $u(x)$ on M_0 is harmonic if and only if $(1+|x|^{2-n})u(x)$ is harmonic on R^n-0. In particular a positive harmonic function u on M_0 has the form:*

$$u(x)=a+\frac{b}{1+|x|^{n-2}}$$

for some constants a, b.

PROOF. From

$$g^{ij}(x)=|x|^4(1+|x|^{n-2})^{-\frac{4}{n-2}}\delta_{ij},\quad g=|x|^{-4n}(1+|x|^{n-2})^{\frac{4n}{n-2}}$$

it follows that

$$\begin{aligned}\Delta u&=-\frac{1}{\sqrt{g}}\sum_{i,j=1}^{n}\frac{\partial}{\partial x^i}\left(\sqrt{g}\,g^{ij}\frac{\partial u}{\partial x^j}\right)\\&=-|x|^4(1+|x|^{n-2})^{-\frac{4}{n-2}}\cdot\left[\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x^{i2}}-\frac{2n-4}{|x|^2(1+|x|^{n-2})}\sum_{i=1}^{n}x^i\frac{\partial u}{\partial x^i}\right]\\&=|x|^{n+2}(1+|x|^{n-2})^{-\frac{n+2}{n-2}}\cdot\Delta_e[(1+|x|^{2-n})u],\end{aligned}$$

where

$$\Delta_e v=-\sum_{i=1}^{n}\frac{\partial^2 v}{\partial x^{i2}}$$

is the Laplace-Beltrami operator for R^n-0. This establishes the first half.

For the second half let u be a positive harmonic function on M. Then by virtue of Bôcher's theorem (cf., e.g., Helms [2, p. 81]), there exists a constant $b\geq 0$ such that the function

$$(1+|x|^{2-n})u(x)-b|x|^{2-n}$$

is harmonic on R^n. Since R^n carries no nonconstant positive harmonic functions and

$$\liminf_{|x|\to\infty}\,[(1+|x|^{2-n})u(x)-b|x|^{2-n}]\geq 0.$$

the desired conclusion follows.

3. For each pair (m, l) of positive integers, m, l, set $k=2^{m-1}(2l-1)-1$. Consider the pair of closed hemispheres

$$H_{ml}=\{8^k x \in M_0 \mid |x|=1 \text{ and } x^1 \geq 0\},$$
$$H'_{ml}=\{8^{-k} x \in M_0 \mid |x|=1 \text{ and } x^1 \geq 0\},$$

where for $a>0$ and $x=(x^1, x^2, \cdots, x^n) \in M_0$, we write $ax=(ax^1, ax^2, \cdots, ax^n)$.

Denote by M_0' the slit manifold obtained from M_0 by deleting all the hemispheres H_{ml} and H_{ml}' $(m, l \geq 1)$.

Take a sequence $\{M_0'(l)\}_1^\infty$ of duplicates of the manifold M_0'. For each fixed $m \geq 1$ and subsequently for fixed $j \geq 0$ and $1 \leq i \leq 2^{m-1}$, connect $M_0'(i+2^m j)$ with $M_0'(i+2^{m-1}+2^m j)$ crosswise along all the hemispheres H_{ml} and $H_{ml}'(l \geq 1)$. The resulting Riemannian n-manifold N is an infinitely sheeted covering manifold of M_0.

We are now ready to state our main result:

THEOREM. *The Riemannian n-manifold N $(n \geq 3)$ carries nonconstant bounded harmonic functions but no nonconstant Dirichlet-finite harmonic functions.*

The proof will be given in 4-5.

4. Let M be the 2-sheeted covering manifold of M_0 obtained by joining, crosswise, two copies of M_0' along all the hemispheres H_{ml} and H'_{ml} $(m, l \geq 1)$. For each $k=2^{m-1}(2l-1)-1$, denote by U_k, U_{-k} the open sets of M which lie over the open annuli $\{x \in M_0 \mid 2^{3k-1}<|x|<2^{3k+1}\}$, $\{x \in M_0 \mid 2^{-3k-1}<|x|<2^{-3k+1}\}$, respectively. Also let S_k, S_{-k} lie over the spheres $\{x \in M_0 \mid |x|=2^{3k}\}$, $\{x \in M_0 \mid |x|=2^{-3k}\}$ respectively. Clearly S_k is a compact subset of U_k, and S_{-k} a compact subset of U_{-k} for all $k=2^{m-1}(2l-1)-1$.

For the families

$H_1(U_k; S_k)=\{u \mid u$ harmonic on U_k with $|u| \leq 1$; u changes sign on $S_k\}$, $H_1(U_{-k}; S_{-k})=\{u \mid u$ harmonic on U_{-k} with $|u| \leq 1$; u changes sign on $S_{-k}\}$,

define
$$q_k=\inf\{q \mid |u(x)| \leq q \text{ for all } x \in S_k \text{ and } u \in H_1(U_k; S_k)\},$$
$$q_{-k}=\inf\{q \mid |u(x)| \leq q \text{ for all } x \in S_{-k} \text{ and } u \in H_1(U_{-k}; S_{-k})\}.$$

It is known that $0<q_k$, $q_{-k}<1$ for all $k=2^{m-1}(2l-1)-1$ (cf., e.g., [Ahlfors-

Sario [1, p. 154]).

LEMMA 2. *There exists a positive constant* $q<1$ *such that* $q\geq q_k$, q_{-k} *for all* $k=2^{m-1}(2l-1)-1$.

PROOF. Let $u\in H_1(U_k;\ S_k)$, $k=2^{m-1}(2l-1)-1$. By Lemma 1 the function $(1+|x|^{2-n})u(x)$ is Δ_e-harmonic (harmonic with respect to the Euclidean structure) on U_k. Since the Δ_e-harmonicity is invariant by the magnification $x=(x^1, \cdots, x^n)\to 8^k x=(8^k x^1, \cdots, 8^k x^n)$, $(1+8^{k(2-n)}|x|^{2-n})u(8^k x)$ is Δ_e-harmonic on U_0. Therefore

$$(1+8^{k(2-n)})\max\{|u(y)|\,|y\in S_k\}\leq q_0'(1+8^{k(2-n)}.2^{n-2})$$

where q_0' is a positive constant with $q_0'<1$ such that

$$\max\{|v(x)|\,|x\in S_0\}\leq q_0'$$

for every Δ_e-harmonic function v on U_0 with $|v|\leq 1$ which changes sign on S_0. Taking the supremum over u and then letting $k\to\infty$, we conclude that

$$\limsup_{k\to\infty} q_k\leq q_0'.$$

For the estimate of q_{-k} for large k, consider $u\in H_1(U_{-k};\ S_{-k})$. As above the function $(1+|x|^{2-n})u(x)$ is Δ_e-harmonic on U_{-k}. In view of the fact that the Δ_e-harmonicity is invariant by the Kelvin transformation, for $\rho=8^{-\frac{k}{2}}$ the function

$$\frac{\rho^{n-2}}{|x|^{n-2}}(1+\rho^{2(2-n)}|x|^{n-2})u\left(\frac{\rho^2}{|x|^2}x\right)$$

is Δ_e-harmonic on U_0. Therefore

$$\limsup_{k\to\infty} q_{-k}\leq q_0'$$

and this completes the proof of Lemma 2.

5. The first half of our theorem is obvious, since the base manifold M_0 carries a nonconstant bounded harmonic function by Lemma 1.

For the second half we shall show that every bounded harmonic function u on N takes the same value at all points in N which lie over the same point of M_0. Then the function u has a finite Dirichlet integral over N onlo when u is a constant.

For each $k=2^{m-1}(2l-1)-1$, consider the open sets v_k, v_{-k} of N which lie over the annuli $\{x\in M_0|2^{3k-1}<|x|<2^{3k+1}\}$, $\{x\in M_0|2^{-3k-1}<|x|<2^{-3k+1}\}$ respectively and the sets $T_k(\subset V_k)$, $T_{-k}(\subset V_{-k})$ over the spheres $\{x\in M_0|\ |x|=2^{3k}\}$, $\{x\in M_0|\ |x|=2^{-3k}\}$ respectively. Each component C of V_k (or V_{-k}) is a duplicate of U_k (or U_{-k}) with $C\cap T_k=S_k$ (or $C\cap T_{-k}=S_{-k}$). Therefore the conclusion of Lemma 2 with the same $q<1$ is applicable in the present situation:

$$\max\{|v(x)|\,|x\in T_k\}\leq q$$

for all $v\in H_1(V_k;\ T_k)$, and this property is true for T_{-k} and V_{-k}. Here we understand that every $v\in H_1(V_k;\ T_k)$ (or $H_1(V_{-k};\ T_{-k})$) is required to change sign on each component of T_k (or T_{-k}).

Let u be a bounded harmonic function on N with $|u|\leq 1$. For each $m\geq 1$, denote by π_m the cover transformation of N which interchanges the sheets of N: points in $M_0'(i+2^m j)$ are interchanged with points, with the same projection, $M_0'(i+2^{m-1}+2^m j)$ for $j\geq 0$ and $1\leq i\leq 2^{m-1}$. Define v_m on N by

$$v_m(x)=\frac{1}{2}[u(x)-u(\pi_m(x))].$$

Clearly v_m is harmonic on N, $|v_m|\leq 1$, and v_m changes sign on each component of T_k and T_{-k} ($k=2^{m-1}(2l-1)-1$ and $l\geq 1$). Therefore

$$\max\{|v_m(x)|\,|x\in T_k\cup T_{-k}\}\leq q$$

for all $l\geq 1$. Since $q^{-1}v_m\in H_1(V_{k'};\ T_{k'})\cap H_1(V_{-k'};\ T_{-k'})$ with $k'=2^{m-1}(2l-3)-1$, $|v_m|\leq q^2$ on $T_{k'}\cup T_{-k'}$. By induction on l, we conclude that $|v_m|\leq q^l$ on $T_{k''}\cup T_{-k''}$, where $k''=2^{m-1}-1$. Letting $l\to\infty$, $|v_m|\equiv 0$ on N, as desired. Therefore

$$u(x)=u(\pi_m(x))$$

on N for all $m\geq 1$, and u attains the same value at all the points in N which lie over the same point in M_0.

COROLLARY. (1) $\dim HB(N)=2$,
(2) $\dim HD(N)=1$.

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