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INTEGRABILITY CONDITIONS OF AN ALMOST CONTACT MANIFOLD

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1. Introduction.

Let V_n be an n-dimensional differentiable manifold. Let there be defined in V_n, a C^∞ vector valued linear function F, a vector field T and a 1-form A satisfying

(1.1) a $$\overline{X}+X=A(X)T,$$

for an arbitrary vector field X, where

(1.1) b $$\overline{X} \overset{\text{def}}{=} F(X).$$

Then V_n is called an almost contact manifold. It can be easily proved that

(1.2) n is odd dimensional$=2m+1$

(1.3) $$\text{rank }(F)=2m,$$

(1.4) $$\overline{T}=0,$$

(1.5) $$A(T)=1,$$

(1.6) $$A(\overline{X})=0,$$

for an arbitrary vector field X.

Agreement (1.1). *In the preceeding and in what follows the equations containing X, Y, Z, U etc. hold for arbitrary vector fields X, Y, Z, U.*

Let there be defined in V_n a metric tensor g satisfying

(1.7) $$g(\overline{X}, \overline{Y})=g(X, Y)-A(X)\ A(Y).$$

Then the almost contact manifold V_n is called an almost Grayan manifold.

Let us put in the almost Grayan manifold V_n

(1.8) $$'F(X, Y)=g(\overline{X}, Y).$$

Then $'F$ is skew symmetric:

(1.9) a $$'F(X, Y)+'F(Y, X)=0$$

and

(1.9) b $$'F(\overline{X}, \overline{Y})='F(X, Y).$$

If in the almost Grayan manifold V_n,

(1.10) a $$'F=dA.$$

Then V_n is called an almost Sasakian manifold. Thus for an almost Sasakian manifold

(1.10) b $\quad 'F(X, Y)=(dA)(X, Y)$

equivalent to

(1.10) c $\quad 'F(X, Y)=(D_XA)(Y)-(D_YA)(X),$

where D is a symmetric connexion.

It is easy to see that for an almost Sasakian manifold

(1.11) $\quad (D_X'F)(Y, Z)+(D_Y'F)(Z, X)+(D_Z'F)(X, Y)=0.$

Nijenhuis tensor N is given by

$$N(X, Y)=[\overline{X}, \overline{Y}]+\overline{\overline{[X, Y]}}-\overline{[\overline{X}, Y]}-\overline{[X, \overline{Y}]}.$$

An almost contact manifold for which

(1.12) $\quad N(X, Y)+(dA)(X, Y)T=0.$

holds is called *an almost contact normal manifold.*

Let us put

(1.13) $\quad l(X) \overset{\text{def}}{=} X-A(X)T,$

(1.14) $\quad m(X)\overset{\text{def}}{=}A(X)T.$

Then

(1.15) $\quad X=l(X)+m(X).$

It can be proved easily that

(1.16) a $\quad \overline{l(X)}=l(\overline{X})=\overline{X},$

(1.16) b $\quad \overline{\overline{l(X)}}=l(\overline{\overline{X}})=-l(X),$

(1.17) $\quad m(\overline{X})=\overline{m(X)}=0,$

(1.18) $\quad l(m(X))=m(l(X))=0,$

(1.19) $\quad l^2(X) \overset{\text{def}}{=} l(l(X))=l(X),$

(1.20) $\quad m^2(X)=m(X)$

(1.21) $\quad l(T)=0, \ m(T)=T.$

The operators l and m applied to the tangent space at each point of the manifold are complementary projection operators. Thus there exist in the manifold two complementary distributions Π_{2m} and Π_1 corresponding to l and m respectively. Π_{2m} is $2m$-dimensional and Π_1 is 1-dimensional.

2. Integrability conditions.

THEOREM (2.1). *The distribution Π_1 is integrable.*

PROOF. The distribution Π_1 is given by

(2.1) a) $X=m(X),$ $\qquad$ b) $l(X)=0.$

In order that $\amalg_1$ is integrable, it is necessary and sufficient that

(2.2) $$(dl)(X, Y)=0$$

be satisfied by (2.1)a. Thus we have

(2.3) $$(dl)(m(X), m(Y))=0.$$

In consequence of (1.18), this equation is equivalent to

(2.4) $$l([m(X), m(Y)])=0$$

In consequence of (1.13) and (1.14) this equation is automatically satisfied. Hence we have the statement.

THEOREM (2.2). *The necessary and sufficient condition that* $\amalg_{2m}$ *be integrable is*

(2.5) a $$A(X)(dA)(T, Y)-A(Y)(dA)(T, X)=(dA)(X, Y),$$

equivalent to

(2.5) b $$(dA)(\overline{X}, \overline{Y})=0.$$

PROOF. The distribution $\amalg_{2m}$ is given by

(2.6) a) $X=l(X)$, b) $m(X)=0$,

In order that $\amalg_{2m}$ is integrable it is necessary and sufficient that

$$(dm)(X, Y)=0$$

be satisfied by (2.6)a. Hence, we have

$$(dm)(l(X), l(Y))=0.$$

In consequence of (1.18), this equation is equivalent to

$$m([l(X), l(Y)])=0.$$

With the help of (1.13) and (1.14) this equation takes the form

$$A(X)\{T(A(Y))-A([T, Y])\}-A(Y)\{T(A(X))-A([T, X])\}$$
$$=X(A(Y))-Y(A(X))-A([X, Y]),$$

which is the equation (2.5)a.

Barring X, Y in (2.5)a and using (1.6), we get (2.5)b.

COROLLARY (2.1). *The equation* (2.5)b *is also equivalent to*

(2.5) c $$A(N(X, Y))=0,$$

or

(2.5) d $$A(N(\overline{X}, \overline{Y}))=0$$

or

(2.5) e $$A(N(\overline{X}, Y))=0.$$

PROOF. In consequence of (1.6), the equation (2.5)b is equivalent to

(2.5) f $$A([\overline{X}, \overline{Y}])=0,$$

which, by virtue of the definition of N is the same as (2.5)c. Barring X and Y

in (2.5)c, we get (2.5)d. Barring X and Y in (2.5)d, we get (2.5)c. We similarly obtain (2.5)e. Barring X in (2.5)e we get (2.5)c. Hence (2.5)c, d, e are equivalent.

THEOREM (2.3). *Necessary and sufficient condition that V_n be integrable is* (2.5).

PROOF. The statement follows from Theos. (2.1) and (2.2) and Cor. (2.1).

COROLLARY (2.2). *If an almost contact manifold V_n is integrable*

$$(dA)(X, Z)(dA)(T, Y)=(dA)(Y, Z)(dA)(T, X).$$

PROOF. The equation follows from (2.5)a, by using the fact that $d^2=0$.

THEOREM (2.4). *The necessary and sufficient condition that an almost contact normal manifold be integrable is*

$$N(X, Y)=A(X)N(T, Y)-A(Y)N(T, X), \tag{2.7 a}$$

equivalent to

$$N(\overline{X}, \overline{Y})=0, \tag{2.7 b}$$

or

$$N(X, \overline{Y})=A(X)N(T, \overline{Y}). \tag{2.7 c}$$

PROOF. Substituting from (1.12) in (2.5)a, b we obtain (2.7)a, b. Barring Y in (2.7)a and using (1.6), we obtain (2.7)c.

THEOREM (2.5). *An almost Sasakian manifold cannot be integrable.*

PROOF. Substituting from (1.10)b in (2.5)a, and using $'F(T, Y)=0$, we get

$$'F=0,$$

equivalent to

$$F=0,$$

which proves the statement.