Kyungpook Math. J.
Volume 13, Number 2
December, 1973

ON H-CURVATURE TENSORS IN KAHLER MANIFOLD

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1. Introduction.

Let us consider a $2n$-dimensional real manifold M_{2n} endowed with a real vector-valued linear function F, satisfying

(1.1) $$\overline{\overline{X}}+X=0$$

for arbitrary vector X, where $\overline{X}=F(X)$. Then F is called an almost complex structure to M_{2n} and M_{2n} is called an 'almost complex manifold'.

An almost complex manifold M_{2n} endowed with a Hermite metric g, such that

(1.2) $$g(\overline{X},\overline{Y})=g(X,Y)$$

is called an 'almost Hermite manifold'.

Let there be a tensor $F(X, Y)$ defined by

(1.3) $$F(X,Y)=g(\overline{X},Y)=-g(X,\overline{Y})$$

which satisfies

(1.3a) $$F(X,\overline{Y})=g(X,Y),$$

(1.3b) $$F(\overline{X},\overline{Y})=F(X,Y),$$

(1.3c) $$F(X,Y)=-F(Y,X).$$

Let D_X be a Riemannian connexion in M_{2n}. An almost Hermite space is called a Kahler manifold if

(1.4) $$(D_XF)(Y)=0$$

Let $K(X,Y,Z)$ be the curvature tensor of type (1,3) of the Kahler manifold. Then

(1.5) $$K(\overline{X},Y,Z)=-K(X,\overline{Y},Z)$$

(1.5a) $$\overline{K(X,Y,Z)}=K(X,Y,\overline{Z}),$$

(1.5b) $$K(X,Y,Z)=-K(Y,X,Z).$$

Let $K(X,Y,Z,T)=g(K(X,Y,Z),T)$ be the curvature tensor of type (0,4) of the Kahler manifold. Then we have

(1.6) $$K(X,Y,Z,T)=K(\overline{X},\overline{Y},Z,T)=K(X,Y,\overline{Z},\overline{T}).$$

If $K(X,Y)$ be the Ricci tensor of the Kahler manifold, then

(1.7) $$K(\overline{X},\overline{Y})=K(X,Y).$$

The vector valued function $K(X)$ defined by

(1.8a) $$g(K(X),Y)=K(X,Y).$$

satisfies

(1.8b) $$XK=2(\text{div } K)\ (X).$$

where K is scalar curvature.

The projective conformal, concircular and conharmonic curvature tensors are respectively given by

(1.10) $$W(X,Y,Z)=K(X,Y,Z)-\frac{1}{2n-1}[XK(Y,Z)-YK(K,Z)],$$

(1.11) $$V(X,Y,Z)=K(X,Y,Z)-\frac{1}{2(n-1)}[XK(Y,Z)-YK(X,Z)-g(X,Z)K(Y)+g(Y,Z)K(X)]+\frac{K}{(2(n-1)(2n-2)}[Xg(Y,Z)-Yg(X,Z)],$$

(1.12) $$C(X,Y,Z)=K(X,Y,Z)-\frac{K}{2n(2n-1)}[Xg(Y,Z)-Yg(X,Z)].$$

and

(1.13) $$L(X,Y,Z)=K(X,Y,Z)-\frac{1}{2(n-1)}[g(Y,Z)K(X)-g(X,Z)K(Y)+XK(Y,Z)-YK(X,Z)].$$

For a Kahler manifold, we have

(1.14a) $$W(\overline{X},\overline{Y},Z,T)=W(X,Y,\overline{Z},\overline{T}),$$

(1.14b) $$V(\overline{X},\overline{Y},Z,T)=V(X,Y,\overline{Z},\overline{T}),$$

(1.14c) $$C(\overline{X},\overline{Y},Z,T)=C(X,Y,\overline{Z},\overline{T}),$$

and

(1.14d) $$L(\overline{X},\overline{Y},Z,T)=L(X,Y,\overline{Z},\overline{T}).$$

where

(1.15a) $$W(X,Y,Z,T)=g(W(X,Y,Z),T),$$

(1.15b) $$V(X,Y,Z,T)=g(V(X,Y,Z),T),$$

(1.15c) $$C(X,Y,Z,T)=g(C(X,Y,Z),T),$$

(1.15d) $$L(X,Y,Z,T)=g(L(X,Y,Z),T).$$

Analogous to projective, conformal concircular and conharmonic curvature tensors, we have the following curvature tensors

(1.16) $$P(X,Y,Z)=K(X,Y,Z)+\frac{1}{(n+1)2}[YK(X,Z)-XK(Y,Z)+\overline{Y}K(\overline{X},Z)-\overline{X}K(\overline{Y},Z)+2\overline{Z}K(\overline{X},Y)],$$

$$(1.17)\quad Q(X,Y,Z)=K(X,Y,Z)+\frac{1}{2(n+2)}[YK(X,Z)-XK(Y,Z)$$
$$+\bar{Y}K(\bar{X},Z)-\bar{X}K(\bar{Y},Z)+2\bar{Z}K(\bar{X},Y)$$
$$+K(Y)g(X,Z)-K(X)g(Y,Z)$$
$$+K(\bar{Y})g(\bar{X},Z)-K(\bar{X})g(\bar{Y},Z)$$
$$+2K(\bar{Z})g(\bar{X},Y)]-\frac{K}{4(n+1)(n+2)}[Yg(X,Z)-Xg(Y,Z)$$
$$+\bar{Y}g(\bar{X},Z)-\bar{X}g(\bar{Y},Z)+2\bar{Z}g(\bar{X},Y)].$$

$$(1.18)\quad R(X,Y,Z)=K(X,Y,Z)+\frac{K}{4n(n+1)}[Yg(X,Z)-Xg(Y,Z)$$
$$+\bar{Y}g(\bar{X},Z)-\bar{X}g(\bar{Y},Z)+2\bar{Z}g(\bar{X},Y)].$$

$$(1.19)\quad S(X,Y,Z)=K(X,Y,Z)+\frac{1}{2(n+2)}[YK(X,Z)-XK(Y,Z)$$
$$+\bar{Y}K(\bar{X},Z)-\bar{X}K(\bar{Y},Z)+X\ 2\bar{Z}K(\bar{X},Y)$$
$$+K(Y)g(X,Z)-K(X)g(Y,Z)$$
$$+K(\bar{Y})g(\bar{X},Z)-K(\bar{X})g(\bar{Y},Z)+2K(\bar{Z})g(\bar{X},Y)].$$

They are called H-projective, H-conformal (Bochner), H-concircular and H-conharmonic curvature tensors.

2. Certain Tensors.

Let us define the tensors of type (1,3)

$$(2.1a)\quad A(X,Y,Z)=XK(Y,Z)+YK(Z,X)+ZK(X,Y),$$
$$(2.1b)\quad B(X,Y,Z)=Xg(Y,Z)+Yg(Z,X)+Zg(X,Y),$$
$$(2.1c)\quad C(X,Y,Z)=K(X)g(Y,Z)+K(Y)g(Z,X)+K(Z)g(X,Y).$$

which are symmetric in X,Y,Z.

If

$$(2.2a)\quad g(A(X,Y,Z),W)=A(X,Y,Z,W),$$
$$(2.2b)\quad g(B(X,Y,Z),W)=B(X,Y,Z,W),$$
$$(2.2c)\quad g(C(X,Y,Z),W)=C(X,Y,Z,W).$$

then

$$(2.3a)\quad A(X,Y,Z,W)=g(X,W)K(Y,Z)+g(Y,W)K(Z,X)+g(Z,W)K(X,Y),$$
$$(2.3b)\quad B(X,Y,Z,W)=g(X,W)g(Y,Z)+g(Y,W)g(Z,X)+g(Z,W)g(X,Y),$$
$$(2.3c)\quad C(X,Y,Z,W)=K(X,W)g(Y,Z)+K(Y,W)g(Z,X)+K(Z,W)g(X,Y).$$

For Einstein manifold $K(X,Y)=\frac{K}{2n}g(X,Y)$

$$(2.4a)\quad A(X,Y,Z,W)=\frac{K}{2n}B(X,Y,Z,W),$$

$$(2.4b)\quad A(X,Y,Z,W)=C(X,Y,Z,W).$$

The expressions for different H-curvature tensors in terms of $A(X,Y,Z)$,

$B(X,Y,Z)$ and $C(X,Y,Z)$ are as follows:

(2.5a) $P(X,Y,Z)=K(X,Y,Z)+\frac{1}{2(n+1)}[A(\bar{X},Y,\bar{Z})-A(X,\bar{Y},\bar{Z})]$.

(2.5b) $Q(X,Y,Z)=K(X,Y,Z)+\frac{1}{2(n+2)}[A(\bar{X},Y,\bar{Z})-A(X,\bar{Y},\bar{Z})$
$+C(\bar{X},Y,\bar{Z})-C(X,\bar{Y},\bar{Z})]+\frac{K}{4(n+1)(n+2)}[B(\bar{X},Y,\bar{Z})$
$-B(X,\bar{Y},\bar{Z})]$,

(2.5c) $R(X,Y,Z)=K(X,Y,Z)+\frac{K}{4(n+1)n}[B(\bar{X},Y,\bar{Z})-B(X,\bar{Y},\bar{Z})]$

(2.5d) $S(X,Y,Z)=K(X,Y,Z)+\frac{1}{2(n+2)}[A(\bar{X},Y,\bar{Z})-A(X,\bar{Y},\bar{Z})$
$+C(\bar{X},Y,\bar{Z})-C(X,\bar{Y},\bar{Z})]$.

For H-projectively flat space

$$K(X,Y,Z)=\frac{k}{4}B(\bar{X},Y,\bar{Z})-B(X,\bar{Y},\bar{Z}))$$

where k is holomorphic sectional curvature such that

$$K=n(n+1)k.$$

Consequently

$$A(X,Y,Z)=\frac{K}{2n}B(X,Y,Z).$$

The following theorems can be proved easily.

THEOREM (2.1). *The H-projective curvature tensor is curvature tensor if and only if $A(X,Y,Z)$ is hybrid in first two slots.*

THEOREM (2.2). *The H-concircular curvature tensor is curvature tensor if and only if either $K=0$ or $B(X,Y,Z)$ is hybrid in the first two slots.*

THEOREM (2.3). *A necessary and sufficient condition that H-conformal curvature tensor coincides with H-conharmonic curvature tensor is that $B(X,Y,Z)$ is hybrid in first two slots.*

THEOREM (2.4). *In Einstein Kahlerian manifold all H-curvature tensors are equal to the curvature tensor if and only if $B(X,Y,Z)$ is hybrid in first two slots or scalar curvature vanishes.*

THEOREM (2.5). *For a Kahlerian manifold, the H-conformal, H-concircular, H-conharmonic and curvature tensors are linearly related as under*

$$(n+2)[Q(X,Y,Z)-S(X,Y,Z)]+n[(K(X,Y,Z)-R(X,Y,Z)]=0.$$

THEOREM (2.6). *For a Kahlerian space the divergence of H-conformal, H-concircular, H-conharmonic curvature tensors and curvature tensor are linearly* *related as under*

$$(n+2)\ [(\text{div}\,Q)\ (X,Y,Z)-(\text{div}\ S)\ (X,Y,S)] \\ +n[(\text{div}\ K)(X,Y,Z)-(\text{div}\ R)\ (X,Y,Z)]=0.$$

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