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SOME BESSEL FUNCTION INTEGRALS

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In an analytic evaluation of definite integrals, one of the most difficult problems is to know where to start. In this paper we suggest a possible solution and illustrate it with a number of examples involving Bessel functions. These considerations were motivated by trying to obtain the Mellin transform of $I_0(ax)-\mathbf{L}_0(ax)$, which occurred in a study of the exchange energy of a dense electron gas in a magnetic field.

Consider the integral transform

$$S\{f(x)\,;y)=\int_0^\infty dx\frac{f(x)}{(x^2+y^2)^{1/2}} \tag{1}$$

We note that the kernel of the Mellin transform[1] x^{s-1}, is merely multiplied by a constant under S:

$$S\{x^{s-1}\,;y\}=2^{-s}B(s,\ \tfrac{1}{2}-\tfrac{1}{2}s)\ y^{s-1}. \tag{2}$$

Also in terms of this transform $I_0(ax)-\mathbf{L}_0(ax)$ has a simple representation:

$$S\{I_0(ax)-\mathbf{L}_0(ax)\,;y\}=\sin ay. \tag{3}$$

This strongly suggests using the Parseval formula for this transform which states

$$\int_0^\infty f(x)S\{g(y)\,:x\}\,dx=\int_0^\infty g(x)S\{f(y)\,;x\}\,dx. \tag{4}$$

Since our considerations are intended to be merely heuristic, we shall not go into the conditions under which (4) is valid, but simply point out that they are satisfied in all the examples considered below.

A short list of transform pairs is given in Table I. It is by no means complete, but merely represents what can be scavenged from the tables of Erdélyi et al[1]. One method for extending it will be presented below. However, by using (4) and Table I we can produce 169 integral identities. For example, using a table of Mellin transforms, we find

$$\int_0^\infty x^{s-1}[I_\nu(ax)-\underset{\sim}{L}_\nu(ax)]\,dx=\pi^{-1}2^{s-1}a^{-s}\;\frac{\Gamma\left(\frac{2+s}{2}\right)\Gamma\left(\frac{1-\nu-s}{2}\right)\Gamma\left(\frac{1+\nu+s}{2}\right)}{\Gamma\left(\nu+1-\frac{s+\nu}{2}\right)} \tag{5}$$

$$\nu>-1,\ -2\nu<\mathrm{Re}\ s<1-\nu$$

and

$$\int_0^\infty x^{s-1}\{Y_\nu(ax)\}^2dx=a^{-s}\,\Gamma(2+s/2)\Big\{2^{s-1}\frac{\Gamma(1-s)}{\Gamma^2(1-s/2)\Gamma(\nu+1-s/2)}$$

$$-\pi^{-5/2}\cos(\pi\nu)\ \Gamma(s/2)\ \Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{s}{2}-\nu\right)\Big\} \tag{6}$$

$$2|\nu|<s<1.$$

These results may be checked by expressing the Bessel functions as Meijer G-functions and using 6.9(14) of Ref. [1]. One other interesting formula, obtained by combining (b) and (k) of Table I, is the Fourier cosine transform

$$\int_0^\infty \cos(2ax)\{[J_\nu\ (bx)]^2+[Y_\nu(bx)]^2\}\,dx=\pi^{-2}\ b^\nu a^{-(\nu+1)}\times$$

$$\times\cos(\pi\nu)\ \Gamma^2\left(\frac{1}{2}+\nu\right)\Gamma^2\left(\frac{1}{2}-\nu\right){}_2F_1\left(\frac{1}{2}+\nu,\ \frac{1}{2}+\nu:1:1-\frac{b^2}{a^2}\right) \tag{7}$$

$$|\nu|<\frac{1}{2}.$$

This does not begin to exhaust the results obtainable from Table I.

One needs to know, in conjunction with this procedure, whether a given function has a simple integral representation of the form (1), that is, what is the inverse transform. Here we give only a heuristic approach to this problem and suggest that the result be checked independently in each case.

By multiplying (1) through by $yJ_0(yz)$ and integrating from 0 to ∞ with respect to y, assuming that the order of integration can be reversed, we obtain

$$\int_0^\infty f(x)\ e^{-xz}\ dx=z^{\frac{1}{2}}\int_0^\infty y^{\frac{1}{2}}\ S\{f(x):y\}\ (yz)^{\frac{1}{2}}\ J_0(yz)\ dy$$

$$\equiv z^{\frac{1}{2}}\ \mathscr{H}_0\left\{y^{\frac{1}{2}}\ S\{f(x):y\}\ :z\right\}.$$

Thus $f(x)$ is given by the inverse Laplace transform of $z^{\frac{1}{2}}$ times the zero order Hankel transform of $y^{\frac{1}{2}}$ times the given function:

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}dz\ e^{zx}\ z^{\frac{1}{2}}\ \mathscr{H}_0\left\{y^{\frac{1}{2}}\ S\{f(x):y\}\ :z\right\} \tag{8}$$

where c is greater than the real part of any singularity of the integrand.

As a rather simple example, suppose we require

$$I=\int_0^\infty \cos(ax)\ J_0[(2bx)^{\frac{1}{2}}]\ K_0[(2bx)^{\frac{1}{2}}]\ dx.$$

We note that[1]

$$\mathscr{K}_0\{y^{\frac{1}{2}}\ J_0[(2by)^{\frac{1}{2}}]\,K_0[(2by)^{\frac{1}{2}}]\ ;\ z\}=\frac{1}{2}z^{-3/2}e^{-b/z}$$

and

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(2z)^{-1}\ e^{zx}\ e^{-b/z}dz=\frac{1}{2}\ J_0\Big(2(bx)^{\frac{1}{2}}\Big).$$

Hence, by (4) and entry (b) of Table I, we have

$$I=\int_0^\infty J_0[2(bx)^{\frac{1}{2}}]\ K_0(ax)\ dx=(\pi/2a)[I_0(b/a)-\underline{L}_0(b/a)], \tag{9}$$

where the integral in (9) is listed in the table of K-transforms in Ref. [1]. The result (9) is also a known cosine transform[1]. In the same way we obtain

$$\int_\lambda^\infty K_0(ax)\Big\{1-\frac{\beta x J_1\ [\beta(x^2-\lambda^2)^{\frac{1}{2}}]}{(x^2-\lambda^2)^{\frac{1}{2}}}\Big\}dx=K_0[\lambda(a^2+\beta^2)^{\frac{1}{2}}], \tag{10}$$

which appears to be new. In deriving (10) we have shown that the inverse S–transform of the quantity in curly brackets (times the approximate step function) in the integrand is

$$(y^2+\lambda^2)^{\frac{1}{2}}\ \exp\{-\beta(y^2+\beta^2)^{\frac{1}{2}}\}.$$

The general procedure suggested by these considerations is the following: Look for an integral representation of a factor in the integrand which has a symmetric kernel. For such integral transforms Parseval's relation (4) is valid, generally under rather mild restrictions. By using this relation the integral may be transformed into a known form or perhaps to one which can be treated by standard methods. Even if the integral cannot ultimately be obtained in closed form, by this means it may be transformed into one which is more amenable to numerical computation. Thus, for example, the integral in Eq. (9) is simpler to calculate than I itself, since the integrand contains only one rather than two oscillating functions and decays as e^{-ax} rather than $e^{-ax^{\frac{1}{2}}}$.

TABLE I

	$f(x)$	$S\{f(x)\,;\,y\}$
(a)	$\sin(ax)$	$(\pi/2)\ [I_0(ay)-\underset{\sim}{L}_0(ay)]$
(b)	$\cos(ax)$	$K_0(ay)$
(c)	$x^{-\frac{1}{2}}\sin(ax)$	$(\pi a/2)^{\frac{1}{2}}\ I_{\frac{1}{4}}\left(\frac{1}{2}ay\right)K_{\frac{1}{4}}\left(\frac{1}{2}ay\right)$
(d)	$x^{-\frac{1}{2}}\cos(ax)$	$(\pi a/2)^{\frac{1}{2}}\ I_{-\frac{1}{4}}\left(\frac{1}{2}ay\right)K_{\frac{1}{4}}\left(\frac{1}{2}ay\right)$
(e)	xe^{-ax}	$(\pi y/2)\ [\underset{\sim}{H}_1(ay)-Y_1(ay)-2/\pi]$
(f)	$xJ_0(ax)$	$a^{-1}\,e^{-ay}$
(g)	$J_\nu(ax)$	$I_{\nu/2}\left(\frac{1}{2}ay\right)K_{\nu/2}\left(\frac{1}{2}ay\right)$, Re $\nu>-1$
(h)	$x^{\nu+1}\,J_\nu(ax)$	$(2/\pi a)^{\frac{1}{2}}\,y^{\nu+\frac{1}{2}}\,K_{\nu+\frac{1}{2}}\,(ay)$, $-1<$ Re $\nu<\frac{1}{2}$
(i)	$x^{1-\nu}\,J_\nu(ax)$	$(\pi/2a)^{\frac{1}{2}}\,y^{\frac{1}{2}-\nu}\ [I_{\nu-\frac{1}{2}}\,(ay)-\underset{\sim}{L}_{\nu-\frac{1}{2}}(ay)]$, Re $\nu>-\frac{1}{2}$
(j)	$Y_\nu\,(ax)$	$-\pi\sec(\pi\nu/2)\ K_{\nu/2}\left(\frac{1}{2}ay\right)\left[K_{\nu/2}\left(\frac{1}{2}ay\right)+\pi\sin(\pi\nu/2)\ I_{\nu/2}\left(\frac{1}{2}ay\right)\right]$ $-1<$ Re $\nu<1$
(k)	$K_\nu(ax)$	$(\pi^2/8)\sec(\pi\nu/2)\left\{\left[J_{\nu/2}\left(\frac{1}{2}ay\right)\right]^2+\left[Y_{\nu/2}\left(\frac{1}{2}ay\right)\right]^2\right\}$ $-1<$ Re $\nu<1$
(l)	$H_\alpha\beta(x)=\begin{cases}1 & \alpha<x<\beta\\ 0 & \text{otherwise}\end{cases}$	$\ln\left(\dfrac{\beta+(\beta^2+y^2)^{\frac{1}{2}}}{\alpha+(\alpha^2+y^2)^{\frac{1}{2}}}\right)$
(m)	x^{s-1}	$\frac{1}{2}\,y^{s-1}\,B\left(s/2,\ \frac{1-s}{2}\right)$, $0<$ Re $s<1$.

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REFERENCE

[1]. A. Erdélyi et al, *Tables of Integral Transforms,* McGraw-Hill, New York 1954, Vol. I, II.