Kyungpook Math. J.
Volume 13, Number 2
December, 1973

SOME COMMUTATIVITY THEOREMS FOR NEAR RINGS

By Steve Ligh

1. Introduction.

Some of the commutativity theorems for rings are the followings:

(A) [6] If R is a ring such that for each x in R, there is an integer $n=n(x)>1$ such that $x^n=x$, then R is commutative.

(B) [6] A ring R is commutative if and only if for each x,y in R, there is an integer $n=n(x)>1$ such that $(xy-yx)^n=xy-yx$.

(C) [7] A finite ring is commutative if and only if all the nilpotent elements are central.

(D) [6] A ring R is commutative if and only if for each x in R, there is an integer $n=n(x)>1$ such that x^n-x is central.

(E) [17] If R is a primary ring with identity which satisfies the identities $(xy)^k=x^ky^k$, $k=n$, $n+1$, $n+2$, where n is a nonnegative integer, then R is commutative.

(F) [19] If R is a ring with identity and $(xy)^2=x^2y^2$, then R is commutative.

Some of the recent work in near rings concern with the extensions of some of the above results to certain classes of near rings. As a by-product, some more elementary proofs of some of the above results were obtained.

The study of commutativity in near rings began in [9] where (A) was extended to d.g. near rings with $n=2$. Then in [1, 11] it was shown that (A) can be extended to d.g. near rings. H.E. Bell showed in [2] that both (B) and (D) are valid for d.g. near rings with identities. We showed in [12] that (B) can be extended to d.g. near rings without assuming the existence of an identity. However, whether one can drop the requirement of an identity in (D) or not is not known. In [12] it was shown that (C) can be extended to finite d.g. near rings with "zero divisors" replacing "nilpotent elements". One of the purposes of this note is to show that both (C) and (D) are valid for distributive near rings and (C) is also valid for d.g. near rings with identities. Theorems (E) and (F) have not been studied if R is a near ring. It will be shown that (E) and (F) are not valid for near rings. En route to these results we will discuss other generalizations related to the above results.

2. Distributive and α-near rings.

A near ring is distributive if each element is right distributive. For some basic definitions and elementary results concerning near rings, see [11].

The following two results will be used frequently.

THEOREM A. [10] *Let R be a distributive near ring and R' its commutator subgroup. Then $R'x=xR'=0$ for each x in R.*

THEOREM B. [13] *Let R be a d.g. near ring. Then R' is an ideal of R and hence R/R' is a ring.*

In [2] Bell showed that both (B) and (D) are valid for d.g. near rings with identities. Later in [12] we proved that (B) is valid for d.g. near rings. Naturally one conjectures that (D) is also valid for d.g. near rings. But this has not been resolved. However, we can show that (D) can be extended to distributive near rings.

THEOREM 1. *Let R be a distributive near ring such that for each x in R, there is an integer $n=n(x)>1$, such that x^n-x is central. Then R is a commutative near ring.*

PROOF. If either $R'=0$ or $R'=R$, then conclusion follows. Suppose $R'\neq 0$. By Theorem B, R/R' is a ring and by (D), R/R' is commutative. Thus $(ab-ba)a=0$ and $a(ab-ba)=0$ imply that $aba=ba^2=a^2b$. Consequently $a^nb=ba^n$ for each positive integer n. Since $(a^n-a)b=b(a^n-a)$, we see that $ab=ba$. Hence R is a commutative near ring.

In [12], (C) was extended to d.g. near rings with "zero divisors" replacing "nilpotent elements". Now we extend (C) to the class of distributive near rings.

THEOREM 2. *Let R be a distributive near ring. Then R is commutative if all nilpotent elements N are central and R/N is finite.*

PROOF. If $N=0$ then by [11, Theorem 4] R is commutative. If $N=R$ then clearly R is commutative. Suppose $N\neq 0$. Since N is an ideal, R/N is commutative by [11, Theorem 4]. Hence for each x in R, there is an $n=n(x)$ such that x^n-x is in N. By Theorem 1, R is commutative.

It is not known whether or not Theorem 2 can be extended to d.g. near rings. However, we can obtain the following result.

THEOREM 3. *Let R be a d.g. near ring with* 1 *and all the nilpotent elements N are central. If R/N is finite, then R is a commutative ring.*

PROOF. If $N=0$, then R is a commutative ring by [11, Theorem 4]. Suppose $N\neq 0$. Since R/N is finite, it follows that x^n-x is in N for each x in R. By Theorem 2 in [2], R is a commutative ring.

Another result in this direction is as follows.

THEOREM 4. *Let R be a distributive near ring such that for every commutator $u=ab-ba$, there is an integer $n=n(u)>1$, such that u^n-u is central. Then every commutator u is central.*

PROOF. If $R'=0$ then R is a ring. In [18] it was shown that Theorem 4 is valid for rings. Hence for every commutator u in R, $(ux-xu)$ is in R' for every x in R. Follow a similar argument as in the proof of Theorem 1, we see that $ux=xu$.

Note that Theorem 4 cannot be extended to near rings in general, neither can one arrive at a sharper conclusion that R is commutative.

Recall [10] that an α-near ring is a d.g. near ring in which the additive inverse of a right distributive element is also right distributive. The following was proved in [10].

THEOREM C. *Let R be an α-near ring. Then $R'x=0$ for each x in R where R' is the commutator subgroup of $(R,+)$.*

Clearly every distributive near ring is an α-near ring but not conversely.

Among the interesting generalizations of Jacobson's "$x^n=x$" theorem is the following result of Herstein [8].

THEOREM D. *If $n>1$ is a fixed positive integer and R is a ring satisfying either $(xy)^n=x^ny^n$ or $(x+y)^n=x^n+y^n$ for all x,y in R, then the commutator ideal of R is nil and the nilpotent elements of R form an ideal.*

Now we consider α-near rings with the above properties.

THEOREM 5. *Let R be an α-near ring and for each x,y in R, either $(xy)^n=x^ny^n$ or $(x+y)^n=x^n+y^n$ for some $n>1$. Then the commutator ideal of R is nil.*

PROOF. By Theorem C, if either $R'=0$ or $R'=R$, then conclusion follows. Suppose $R'\neq 0$. Since R' is an ideal and R/R' is a ring, by Theorem D, commutators of R/R' are nilpotent. Hence there is an integer k such that $(xy-yx)^k$ is in R'. Again by Theorem C, $(xy-yx)^{k+1}=0$.

3. Other commutativity theorems.

The following theorem was proved in [19].

THEOREM E. *Let R be a ring with* 1 *and $(xy)^2=x^2y^2$ for each x,y in R. Then R is commutative.*

The above result is not valid for near rings [4, p. 256]. In this example, $(R,+)$ is not abelian. The following result is in this direction.

THEOREM 6. *Let R be a near ring with* 1 *and $(xy)^2=x^2y^2$. If R has no nonzero nilpotent elements, then $(R,+)$ is abelian.*

PROOF. First we show that $0x=0$ for each x in R. Since $(0x)^2=00xx$, it follows that $0x=0xx$. Now $0(1+x)=0(1+x)(1+x)=0((1+x)+(1+x)x)$. Hence $0(1+x)x=0$ imply that $0x=0$. By Lemma 3 in [1] R is a subdirect sum of near rings R_i without zero divisors. Since homomorphic image of R inherits the property that $(xy)^2=x^2y^2$, we see that $x(-1)x(-1)=xx$ implies that $(-1)x(-1)=x$ for each x in R_i. Thus $x(-1)=(-1)x$ and this implies that $(R_i,+)$ is abelian. Consequently $(R,+)$ is abelian.

We conjecture that R is a commutative ring in Theorem 6.

We mention another result related to Theorem E.

THEOREM F. [17] *Let R be a ring with* 1 *and suppose further that R is primary. If $(xy)^n=x^ny^n$ for three consecutive integers, then R is commutative.*

The examples in [3] show that Theorem F is not valid for near rings. The case whether Theorem F is true for arbitrary rings has not yet been resolved.

Another special class of near rings in this direction is the class of near rings which satisfy the (xyz) property. That is, a near ring R has the (xyz) property if $xyz=yxz$ for each x,y,z in R. This class of near rings has been studied in [14, 15, 20]. Observe that a near ring R with the (xyz) property satisfies $(ab)^2=a^2b^2$ for each a,b in R. From Theorem D, we see that if a ring R has the (xyz) property and no nonzero nilpotent elements, then R is a commutative ring. It is not true for near rings since the boolean near rings defined in [5] have the (xyz) property and $x^2=x$ for each x in the near ring. On the other hand, a ring having the property that $(xy)^2=x^2y^2$ might not have the (xyz) property. This is illustrated by the two non-commutative rings defined on the Klein group.

4. Concluding remarks.

As evident so far in our discussion, every commutativity theorem for near rings comes from those of rings. A commutativity theorem for general near rings was given in [14]. It is hoped that more such result can be obtained. It is also evident that investigation on near rings with the (xyz) property or "$(xy)^k=x^ky^k$" property should produce interesting results.

University of Southwestern Louisiana
Lafayette, Louisiana 70501

REFERENCES

[1] H.E. Bell, *Near rings in which each element is a power of itself,* Bull. Austral. Math. Soc. 2(1970), 363—368.

[2] H.E. Bell, *Certain near rings are rings,* J. London Math. Soc. (2), 4(1971) 264—270.

[3] J.R. Clay, *The near rings on groups of low order,* Math. Z. 104(1968), 364—371.

[4] J.R. Clay, *Research in near ring theory using a digital computer,* BIT, 10(1970), 249—265.

[5] J.R. Clay and D.A. Lawver, *Boolean near rings,* Canad. Math. Bull. 12(1969), 265—274.

[6] I.N. Herstein, Noncommutative rings, John Wiley and Sons, Inc. 1968.

[7] I.N. Herstein, *A note on rings with central nilpotent elements,* Proc. Amer. Math. Soc. 5(1954), 620.

[8] I.N. Herstein, *Power maps in rings,* Michigan Math. J. 8(1961), 29—32.

[9] Steve Ligh, *On boolean near rings,* Bull. Austral. Math. Soc. 1(1969), 375—379.

[10] Steve Ligh, *On the commutativity of near rings,* Kyungpook Math. J. 10(1970), 105—106.

[11] Steve Ligh, *On the commutativity of near rings,* II., Kyungpook Math. J. 11(1971), 159—163.

[12] Steve Ligh, *On the commutativity of near rings,* III., Bull. Austral. Math. Soc. 6 (1972), 459—464.

[13] Steve Ligh, *On the additive group of finite near integral domains and simple d.g. near rings,* Monatsh. Math. (to appear)

[14] Steve Ligh, *The structure of a special class of near rings*, J. Austral. Math. Soc. 13 (1972), 141—146.

[15] Steve Ligh, *A special class of near rings*, submitted.

[16] Steve Ligh, *The structure of certain classes of rings and near rings*, submitted.

[17] J. LUh, *A commutativity theorem for primary rings*, Acta Math. Scient. Hungaricae, 22(1971), 211—213.

[18] W.S. Martindale, Ⅲ., *The commutativity of a special class of rings*, Canad. J. Math. 12(1960), 263—268.

[19] E.C. Johnsen, D.C. Outcalt and A. Yaqub, *An elementary commutativity theorem for rings*, Amer. Math. Monthly, 75(1968), 288—289.

[20] G. Szeto, *On a class of near rings*, J. Austral. Math. Soc. 14(1972), 17—19.