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A NOTE ON WEAKLY COMPACTLY GENERATED BANACH SPACES WHICH ARE SEPARABLE

By Joe Howard and Kenneth Melendez

It is well known that a Banach space X is separable if and only if it is norm compactly generated, i.e. there is a compact set in X which is fundamental. Since reflexive (even quasireflexive) spaces are weakly compactly generated (WCG), a non-separable reflexive space is WCG but not separable. It is natural to ask, under what condition, or conditions, is a WCG space separable?

We use the following properties to answer the above question.

(a) A Banach space will be called semiseparable if its conjugate X' contains a sequence $\{f_n\}$ total over X.

(b) A Banach space X is almost reflexive if every bounded sequence in X contains a weak Cauchy subsequence.

(c) X has the DP (weak* DP) property if for every weakly convergent to 0 sequence $\{x_n\}$ in X and for every weakly (weak*) convergent to 0 sequence $\{x'_n\}$ in X', $\lim_n x'_n x_n=0$.

If X is separable, then both X and X' are semiseparable. The non-separable space l_∞ of bounded sequences is, therefore, semiseparable. Almost reflexive spaces include those spaces which have separable duals. Spaces with the DP (Dunford-Pettis) property are well known [2]. Infinite dimensional reflexive spaces do not have the DP property. Examples of spaces which have the weak* DP property are $L_\infty(S,\Sigma,\mu)$, l_1, and the Grothendieck space l_∞. Spaces which do not have the weak* DP property include c_0 and $C[0, 1]$. If X is infinite dimensional and has the weak* DP property, then X' is not separable.

THEOREM 1. *A Banach space X is WCG and semiseparable if and only if X is separable.*

PROOF. Let X be WCG and semiseparable, and let K be weakly compact and fundamental in X. Then by the lemma in [6], K is metrizable in the weak topology. Thus X is separable in the weak topology, so separable in the norm topology.

COROLLARY 2. *Let K be weakly compact and fundamental in X. If $C(K)$ is semiseparable, then both $C(K)$ and X are separable.*

PROOF. $C(K)$ is *WCG* [1], so is separable by Theorem 1. Therefore, K is metrizable, and X is separable.

COROLLARY 3. *Let K be weakly compact and fundamental in X. If $C(K)$ is isomorphic to a conjugate Banach space, then both $C(K)$ and X are separable.*

PROOF. $C(K)$ is *WCG* and has the *DP* property [2]. By Theorem 2.1 of [5] and Eberlain's Theorem, $C(K)$ is separable. So K is metrizable and X is separable.

THEOREM 4. *Let X have weak* DP property. Then X is WCG if and only if X is separable.*

PROOF. Assume X is *WCG*. Then the unit cell of X' is weak* sequentially compact [1]. Since X has the weak* *DP* property, every weak Cauchy sequence in X converges in the norm topology of X, so by Eberlain's Theorem, every weakly compact subset of X is norm compact. Since X is *WCG*, X must be separable.

NOTE. There is no relation between weak* *DP* and semiseparable properties.

COROLLARY 5. *Suppose X has DP property and X' is almost reflexive. Then X is WCG if and only if X is separable.*

PROOF. If X has the *DP* property and X' almost reflexive, then a sequence in X is weak Cauchy if and only if it is norm Cauchy [4]. Therefore X has the weak* *DP* property.

In Corollary 5, we cannot drop X' being almost reflexive, for there is a space $C(K)$ which is *WCG* and has *DP* property, but is not separable.

We next consider conjugate *WCG* Banach spaces. Lindenstrauss [3] gave the following proposition.

PROPOSITION 6. *Let X be a separable Banach space. Then X' is WCG if and only if X' is separable.*

THEOREM 7. *Let X have DP property. Then X' is WCG if and only if X' is separable.*

PROOF. Let X' be WCG. Then the unit sphere of X'' is weak* sequentially compact, and hence X is almost reflexive. Now by [4], a sequence in X' is weak Cauchy if and only if it is norm Cauchy. By Eberlain's Theorem, every weakly compact subset of X' is compact and thus separable. Since X' is WCG, X' is separable.

REMARK. Rosenthal [5] has conjectured that if X has DP property and is isomorphic to a subspace of a WCG conjugate Banach space, then X is separable.

COROLLARY 8. *Let K be weakly compact and fundamental in X. If $C(K)'$ is WCG, then both $C(K)'$ and X are separable.*

PROOF. Since $C(K)$ has the DP property, $C(K)'$ is separable by Theorem 7. Hence $C(K)$ is separable, so X is also separable.

In bringing several properties together which have previously been considered, a nice conclusion unfolds.

THEOREM 9. *If X is a WCG conjugate Banach space, then X has no infinite dimensional subspace isomorphic to an almost reflexive Banach space with the DP property. In particular, X has no subspace isomorphic to c_0.*

PROOF. Let Y be an almost reflexive Banach space with the DP property. If Y is isomorphic to a subspace of X, then by [5] every weak Cauchy sequence in Y converges in the norm topology on Y. If $\{y_n\}$ is a sequence in the unit disk of Y, then $\{y_n\}$ has a weak Cauchy subsequence $\{y_{n_i}\}$ since Y is almost reflexive. Thus $\{y_{n_i}\}$ converges in the norm topology and it follows that the unit disk of Y is compact. Hence Y is finite dimensional; a contradiction.

COROLLARY 10. *If X' is a WCG conjugate Banach space with the DP property, then X'' is not WCG.*

PROOF. If X'' is WCG, then X'' has a weak* sequentially compact unit disk, and therefore X' would be almost reflexive. This contradicts Theorem 9.

Oklahoma State University
Stilwater, Oklahoma
U.S.A.

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