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COMMON FIXED POINT THEOREM FOR SOME BOUNDED LINEAR OPERATORS

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1. Preliminary and notation.

In [3], Markov and Kakutani had found what families of bounded linear operators have common fixed point property in a compact convex subset of a linear topological space.

In this paper, we shall find what families of bouned linear operators have common fixed point property in a weakly compact convex subset of a Banach space. First, we can easily see that the main theorem in [1] is the following form for bounded linear operator:

THEOREM. *Let K be a weakly compact convex subset of a B-space. If $T: K \to K$ is a linear operator* with $\|T\| \leq 1$, *then T has a fixed point in K.*

On the basis of this theorem, we shall find the condition that some families of bounded linear operators has common fixed point in K.

We consider $\mathscr{X}$ as a B-space with norm $\|\cdot\|$ and d as a metric in $\mathscr{X}$ by $d=\|\cdot\|$. If $\mathscr{F}$ is a family of mappings on $\mathscr{X}$, for each $x \in \mathscr{X}$, we denote $\mathscr{F}(x)=\{Tx: T\in\mathscr{F}\}$. If $A\subset\mathscr{X}$ is nonempty, $d(A)$ denotes the diameter of A with respect to d. If (*) is property, then a mapping T or a family $\mathscr{F}$ of mappings on $\mathscr{X}$ is said to have (*) iff T or $\mathscr{F}$ has (*) at x for each $x\in\mathscr{X}$, $\mathscr{F}$ has (*) iff each $T\in\mathscr{F}$ has (*). If $A\subset\mathscr{X}$ is nonempty, we denote by $\mathrm{Co}(A)$ the convex hull of A and $\overline{\mathrm{Co}}(A)$ the closed convex hull of A.

Unless otherwise, we shall follow the terminology and definition in [3]

2. Common fixed point theorem.

DEFINITION 2.1. Let $K\subset\mathscr{X}$ be nonempty closed convex and $\mathscr{F}$ be a semigroup with identity of mappings on K. For each $x\in K$, let $C(\mathscr{F}, x)$ be the smallest closed convex subset of K containing x which is $\mathscr{F}$-invariant (i.e. invariant under each $T\in\mathscr{F}$).

Then (i) $\mathscr{F}$ is said to have *convex diminishing orbital diameters* (c.d.o.d.) at $x \in K$ iff either $\mathscr{F}(x)=\{x\}$ or there exist an $x_0 \in \mathrm{Co}(\mathscr{F}(x))$ such that $d(\mathscr{F}(x_0)) < d(\mathscr{F}(x)) < \infty$.

(ii) $\mathscr{F}$ is said to have *regular orbital diameter* (r.o.d.) at $x \in K$ iff either $\mathscr{F}(x)=\{x\}$ or there exist $y, z \in C(\mathscr{F}, x)$ such that $\sup\{\|y-Tz\| : T \in \mathscr{F}\} < d(C(\mathscr{F}, x))$.

THEOREM 2.1. *Let K be a nonempty weakly compact convex subset of $\mathscr{X}$ and $\mathscr{F}$ be a family with identity of linear operators which map K into itself satisfying the following properties*: (1) *$T\mathscr{F}=\mathscr{F}$ and $\|T\| \leq 1$ for each $T \in \mathscr{F}$*, (2) *$\mathscr{F}$ has r.o.d. in K. Then $\mathscr{F}$ has a common fixed point in K.*

PROOF. By weak compactness of K and by Zorn's lemma, let K_1 be minimal with respect to being a nonempty closed convex subset of K which is $\mathscr{F}$-invariant. Suppose there exist an $x_0 \in K$ and a $T \in \mathscr{F}$ such that $T(x_0) \neq x_0$.

By minimality of K_1, we must have $C(\mathscr{F}, x_0)=K$. Since $\mathscr{F}$ has r.o.d. and $F(x_0) \neq \{x_0\}$, there exist y_0, $z_0 \in C(F, x_0)$ with $r_0=\sup\{\|y_0-Tz_0\| : T \in \mathscr{F}\} < d(C(\mathscr{F}, x_0))$.

Define $M=\{y \in K_1 : \sup\{\|y-T(z_0)\| : T \in \mathscr{F}\} \leq r_0\}$.

Then M is nonempty as $y \in M$. It is clear that M is also closed and convex. We shall show that M is $\mathscr{F}$-invariant. Indeed let $T_1 \in \mathscr{F}$ and $y \in M$. Then $\sup\{\|T_1y-Tz_0\| : T \in \mathscr{F}\} = \sup\{\|T_1y-T_1T'z_0\| : TT'=T,\ T \in \mathscr{F}\} \leq \sup\{\|T_1\|\|y-T'z_0\| : TT'=T,\ T \in \mathscr{F}\} = \|T_1\| \sup\{\|y-T'z_0\| : TT'=T,\ T \in \mathscr{F}\} \leq \sup\{\|y-T'z_0\| : TT'=T,\ T \in \mathscr{F}\} \leq r_0$. Hence $Ty \in M$, i.e. M is $\mathscr{F}$-invauiant. Hence $M=K$, by minimality of K. Next we define $N=\{z \in K_1 : \|y-Tz\| \leq r_0$, for all $y \in K_1$, all $T \in \mathscr{F}\}$. Then N is nonempty as $z \in N$. Since each $T \in \mathscr{F}$ is continuous, N is closed. Convexity is clear. We shall show that N is $\mathscr{F}$-invariant. Indeed let $T_1 \in \mathscr{F}$ and $z \in N$. Then $\|y-T(T_1z)\| = \|y-(TT_1)z\| \leq r_0$, for all $y \in K_1$ and all $T \in \mathscr{F}$, since $TT_1 \in \mathscr{F}$. Hence $T_1z \in N$, i.e. N is $\mathscr{F}$-invariant. By minimality of K_1 again, $K_1=N$. Since $r_0 < d(C(\mathscr{F}, x_0)=d(K_1)$, there are $a, b \in K_1$ with $\|a-b\| > r_0$. Since $I \in \mathscr{F}$, it follows that neither a nor b is in N, which is contradiction. Therefore $T(x)=x$ for each $x \in K_1$ and each $T \in \mathscr{F}$. Since K is nonempty, $\mathscr{F}$ has a common fixed point.

DDEFINITION 2.2. $K \subset \mathscr{X}$ is said to have *normal structure* iff for any bounded convex subset H of K, if H contains more than one point, then there exists an

$x_0 \in H$ such that sup $\{\|x-x_0\| : x \in H\} < d(H)$.

COROLLARY 2.2. *Let $K \subset \mathscr{X}$ be nonempty weakly compact convex and $\mathscr{F}$ be a family with identity of linear operators which map K into itself satisfying the following properties*: (1) $T\mathscr{F}=\mathscr{F}$ *and* $\|T\| \leq 1$ *for each* $T \in \mathscr{F}$. (2) $\overline{\mathrm{Co}}(\mathscr{F}(x))$ *has normal structure for each* $x \in K$. *Then* $\mathscr{F}$ *has a common fixed point in K.*

PROOF. We can easily see that $\overline{\mathrm{Co}}\,(\mathscr{F}(x))$ has normal structure implies $\mathscr{F}$ has r.o.d. at x. Hence by theorem 2.1. above result follows.

COROLLARY 2.3. *Let $K \subset \mathscr{X}$ be nonempty weakly compact convex with normal structure and $\mathscr{F}$ be a family with idenity of linear operators which map K into itself satisfying* $T\mathscr{F}=\mathscr{F}$ *and* $\|T\| \leq 1$ *for each* $T \in \mathscr{F}$. *Then* $\mathscr{F}$ *has common fixed point in K.*

COROLLARY 2.4. *Let $K \subset \mathscr{X}$ be nonempty weakly compact convex and $\mathscr{F}$ be a family with identity of linear operators which map K into itself satisfying the following properties*: (1) $T\mathscr{F}=\mathscr{F}$ *and* $\|T\| \leq 1$ *for each* $T \in \mathscr{F}$. (2) $\mathscr{F}$ *has c.d.o.d. in K. Then* $\mathscr{F}$ *has common fixed point in K.*

PROOF. It is clear that $\mathscr{F}$ has c.d.o.d. at x implies $\mathscr{F}$ has r.o.d. at x.

REFERENCES

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