Kyungpook Math. J.
Volume 13, Number 2
December, 1973

PROBLEM OF EFFICIENCY OF CROSS-SECTION OF AN OPEN CHANNEL

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Introduction.

To the Civil Engineer the most important property of an open channel is its capacity to carry water. The discharge in an open channel is dependent on its cross-sectional area and the shape of the wetted perimeter of its cross-section as has been shown by Sellin (1969) by using the empirical formula of Manning (1891). Here flow in the channel has been assumed to be uniform meaning thereby that the size and shape of the cross-section are constant along the length of the channel. In actual practice, uniform flow will be established in any channel which continues sufficiently far with a constant slope and cross-section. The problem of maximum discharge in an open channel when its cross-sectional area is given and the wetted perimeter is subjected to certain limitations but without any restrictions on the width of the free surface, has been dealt by Sellin (1969) for several special cases. In the present paper, the author studies the problem of maximum discharge in an open channel when the width of the free surface and the cross-sectional area of the channel are given but no limitation is imposed on shape of the wetted perimeter of the cross-section. It has been found that for a given width of the free surface and a given cross-sectional area of the channel, maximum discharge will occur if the wetted perimeter of the cross-section is an arc of a circle. The circle has been completely determined. It is suggested that the result derived here could be of vital importance in the designs of drainage, flood discharge channels and irrigation canals.

Mathematical formulation of the problem.

Consider steady uniform incompressible viscous flow in an open channel. Let A be the cross-sectional area of the channel and let 21 be the width of its free surface. The shape of the wetted perimeter of the cross-section is not specified. Take x-axis along the width of the free surface of the channel and y-axis along a central line perpendicular to the x-axis and pointing downward. We wish to determine the shape of the wetted perimeter in order that the discharge may be maximum. Suppose that the shape of the wetted perimeter is given by the equation

$$y=f(x). \tag{1}$$

Daugherty and Franzini (1965) have shown on the basis of empirical formulae that the discharge in an open channel will be maximum for a given cross-sectional area when the wetted perimeter is minimum. Hence the present problem reduces to determining $y=f(x)$ subject to the following conditions:

$$\int_{-l}^{l} y dx=A, \tag{2}$$

$$y=0 \text{ at } x=-l \text{ and } y=0 \text{ at } x=l, \tag{3}$$

$$\int_{-l}^{l} \sqrt{1+y'^2}\ dx \text{ is minimum.} \tag{4}$$

Here y' means the derivative of y with respect to y.

Now we shall rewrite eqns. (2), (3), (4) in dimensionless form. For this, we take a characteristic length 1.

Introducing

$$\xi=x/l \text{ and } \eta=y/l \tag{5}$$

eqns. (2), (3), (4) after simplification become

$$\int_{-1}^{1} \eta d\xi=A/l^2, \tag{6}$$

$$\eta=0 \text{ at } \xi=-1 \text{ and } \eta=0 \text{ at } \xi=1, \tag{7}$$

$$\int_{-1}^{1} \sqrt{1+\eta'^2}\ d\xi \text{ is minimum.} \tag{8}$$

Here η' means the derivative of η with respect to ξ.

Solution.

We shall apply the variational procedure outlined by Hildebrand (1968) to determine η satisfying the conditions (6), (7), (8). To do this, we let

$$H=\sqrt{1+\eta'^2}+\lambda\eta, \tag{9}$$

where λ is the Lagrange multiplier.

The Euler equation corresponding to the minimization of the integral of $H=\sqrt{1+\eta'^2}+\lambda\eta$ is

$$\frac{d}{d\xi}\left[\frac{\eta'}{\sqrt{1+\eta'^2}}\right]-\lambda=0 \tag{10}$$

Integrating (10), we get

$$\frac{\eta'}{\sqrt{1+\eta'^2}}-\lambda\xi=c_1,$$

where c_1 is an arbitrary constant

or
$$\eta'=\pm\frac{(\lambda\xi+c_1)}{\sqrt{1-(\lambda\xi+c_1)^2}}. \tag{11}$$

Integrating again and re-arranging the terms, we get

$$\eta=\pm\sqrt{\frac{1}{\lambda^2}-\left(\xi+\frac{c_1}{\lambda}\right)^2}+c_2, \tag{12}$$

where c_2 is another arbitrary constant.

Now we shall determine c_1, c_2 and λ. Applying the conditions (7), (12) yields

$$0=\pm\sqrt{\frac{1}{\lambda^2}-\left(-1+\frac{c_1}{\lambda}\right)^2} \tag{13}$$

$$0=\pm\sqrt{\frac{1}{\lambda^2}-\left(1+\frac{c_1}{\lambda}\right)^2} \tag{14}$$

Eqns. (13) and (14) give

$$c_1=0 \text{ and } c_2=\mp\sqrt{\frac{1}{\lambda_2}-1}$$

Substituting these values of c_1 and c_2 in (12), we get after some simplification

$$\xi^2+\left(\eta\pm\sqrt{\frac{1}{\lambda^2}-1}\right)^2=\frac{1}{\lambda^2} \tag{15}$$

This eqn. (15) shows that the wetted perimeter of the cross-section of the channel is bounded by a circular arc whose centre lies on the η-axis. Further, we notice that if $A/l^2\leq\pi/2$, the wetted perimeter is bounded by a minor arc of the circle

$$\xi^2+\left(\eta+\sqrt{\frac{1}{\lambda^2}-1}\right)^2=\frac{1}{\lambda^2}, \tag{16}$$

where λ is given by the equation

$$\frac{1}{\lambda^2}\sin^{-1}\lambda-\sqrt{\frac{1}{\lambda^2}-1}=A/l^2 \tag{17}$$

This eqn. (17) has been obtained by applying the condition (6) which states that the area bounded by the wetted perimeter and the free surface is A/l^2.

And if $A/l^2\geq\frac{\pi}{2}$, the wetted perimeter is bounded by a major arc of the circle

$$\xi^2+\left(\eta-\sqrt{\frac{1}{\lambda^2}-1}\right)^2=\frac{1}{\lambda^2}, \tag{18}$$

where λ is given by the equation

$$\frac{\pi-\sin^{-1}\lambda}{\lambda^2}+\sqrt{\frac{1}{\lambda^2}-1}=A/l^2 \tag{19}$$

In particular, if $A/l^2=\frac{\pi}{2}$, eqns. (17) and (19) both give $\lambda=1$. And hence in this case, the wetted perimeter is bounded by a semicircular arc whose equation is

$$\xi^2+\eta^2=1 \ (\eta\geq 0) \tag{20}$$

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