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A POSITIVE LINEAR FUNCTIONAL ON AN ORDERED TOPOLOGICAL VECTOR LATTICE

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The basic properties of the I-summable class $L^1(I)$ on an ordered topological vector lattice has been studied by H. Anton and W.J. Pervin. Extending their methods in this paper, we are going to give some fundamental theorems which are the analogues for the positive functional I of Fatou's Lemma and the Lebesgue convergence theorem for the ordered topological vector lattice in §2, and considered the uniqueness of I in §3.

Finally we introduce two kinds of measurability on the ordered topological vector lattice and then establish an implicational relation between them.

§1. Preliminaries and notations.

We shall introduce terminologies and notations. Let E be a fixed ordered topological vector lattice with the real field throughout this paper. A non empty subset L of E is called *integration lattice* if L is a vector lattice and for every $a \in E$ there exists an element $\hat{a} \in L$ such that $a \leq \hat{a}$. If L is an integration lattice, a positive linear functional $I: L \to R$ is called an *elementary integral* if $I(a_n) \to 0$ whenever $\{a_n\}$ is a sequence in L such that $a_n \downarrow 0$ with respect to the vector space topology. For simplicity we will assume that L is a fixed integration lattice on an ordered topological vector lattice E, and that I is an elementary integral on L. The positive cone of E will be denoted by P. A point $a \in E$ will be called an *upper element* if there exists an sequence $\{a_n\}$ in L such that $a_n \uparrow a$. The class of all upper elements will be denoted by U. From the continuity of the linear and lattice operations it is obvious that the class U is a vector lattice and that I is well-defined on U. We can extend I from L to U by defining $I(x) = \lim_{n\to\infty} I(x_n)$, where $\{x_n\}$ is any sequence in L such that $x_n \uparrow x$.

It follows from the second condition of the integral lattice that I is finite valued. It is also easy to see that $I(a+b) = I(a) + I(b)$ and $I(ka) = kI(a)$ for real $k \geq 0$ and a, b in U. Here we shall prepare the following theorem which has been proved by H. Anton and W.J. Pervin [1].

THEOREM 1.1. (i) *I is monotone and strictly positive on U.*
(ii) *If $\{a_n\}$ is a sequence in U such that $a_n \uparrow a$, then $a \in U$ and $I(a_n) \to I(a)$.*

Now define $-U$ by $-U=\{x \in E \mid -x \in U\}$. It is easy to show that $-U$ is a vector lattice and that $x \in U$ iff there exists $\{x_n\}$ in L such that $x_n \downarrow x$. If $a \in -U$, we define $I(a)$ by $I(a)=-I(-a)$. This definition extends I as a monotone and strictly positive functional from U to $U \cup -U$ and for $a, b \in -U$ and real $c \geq 0$, we have $I(a+b)=I(a)+I(b)$ and $I(ca)=cI(a)$, further, if $a_n \in -U$ and $a_n \downarrow a$, then $a \in -U$ and $I(a_n) \to I(a)$. H. Anton and W.J. Pervin also introduced the following definition in [1].

DEFINITION. 1.2. An element a in the topological vector lattice E is said to be *I-summable* if given any $\varepsilon > 0$ there exists a pair of elements $y \in -U$ and $z \in U$ such that $y \leq a \leq z$ with $I(z)-I(y) < \varepsilon$.

The class of I-summable elements will be denoted by $L^1(I)$, the following remarkable fundamental results for the I-summable class $L^1(I)$ has been proved by H. Anton and W.J. Pervin [1].

THEOREM 1.3. (i) *If $a \in L^1(I)$ then* $\sup\{I(y) \mid y \in -U,\ y \leq a\} = \inf\{I(z) \mid z \in U,\ a \leq z\}$.

(ii) *For each $a \in L^1(I)$, if we put $J(a)=\sup\{I(y) \mid y \in U,\ y \leq a\}$ then $J(a)=I(a)$ for each $a \in U \cup -U$.*

(iii) *$L \subset U \subset E$, and $L \subset L^1(I)$*

(iv) *$L^1(I)$ is an integration lattice and I is a strictly positive functional on $L^1(I)$.*

DEFINITION. 1.4. A topological vector lattice is said to have the *monotone convergence property* if every monotone increasing sequence which is bounded above converges.

THEOREM. 1.5 (Monotone Convergence Thoerem). *If E is a topological vector lattice with the monotone convergence property, and if $\{a_n\}$ is a sequence in $L^1(I)$ such that $a_n \uparrow a$, then $a \in L^1(I)$ and $I(a_n) \to I(a)$.*

COROLLARY. 1.6. *If E is a topological vector lattice with the monotone convergence property, and if $\{a_n\}$ is a sequence in $L^1(I)$ such that $a_n \to a$ then $a \in L^1(I)$ and $I(a_n) \to I(a)$.*

COROLLARY. 1.7. *$L^1(I)$ is a complete lattice.*

§ 2. Sums and Fatou's lemma in $L^1(I)$.

By using the linear, topological, and lattice structure of in the Daniell method of integration for real valued functions on a set Ω, a I-summable theory for topological vector lattice will be developed in this section. From now on we consider the ordered topological vector lattice E with the monotone convergence property. An open problem is to characterize those ordered topological vector lattice with the monotone convergence property.

LEMMA. 2.1. *An element e in P belong to U if and only if there is a sequence $\{a_n\}$ in $P \cap U$ such that $e=\sum_{n=1}^{\infty} a_n$. In this case $I(e)=\sum_{n=1}^{\infty} I(a_n)$.*

PROOF. If we set $\sum_{k=1}^{n} a_k=b_n$, then $b_n \in L$ for $k=1, 2, \cdots, n$, since L is a linear subspace of E, and $b_n \uparrow e$, by the Theorem 1.5. So the "if" part is trivial. On the other hand, let e be in P and $b_n \uparrow e$ with $b_n \in L$. Without loss of generality we may assume that each b_n is in P by setting b_n by $b_n \vee 0$. Set $a_1=b_1$, $a_n=b_n-b_{n-1}$ for $n>1$. Then $e=\sum_{n=1}^{\infty} a_n$, and by the Monotone Convergence Theorem we have $I(e)=\lim_{n\to\infty} I(b_n)=\lim_{n\to\infty} I(\sum_{k=1}^{n} a_k)=\lim_{n\to\infty} \sum_{k=1}^{n} I(a_k)=\sum_{k=1}^{\infty} I(a_k)$.

For an arbitrary element e in E we define the upper elementary integral $\bar{I}$ by setting $\bar{I}(e)=\inf\{I(a) \mid a \geq e,\ a \in U\}$, where we adopt the convention that the infimum of the empty set is $+\infty$. We define the lower elementary integral $\underline{I}$ by setting $\underline{I}(e)= \sup\{I(a) \mid a \leq e,\ a \in -U\}$. The following Lemma follows directly from the definition of $\bar{I}$.

LEMMA. 2.2. *Let $\{e_n\}$ be a sequence in P, and let $e=\sum_{n=1}^{\infty} e_n$. Then $\bar{I}(e) \leq \sum_{n=1}^{\infty} \bar{I}(e_n)$.*

PROOF. If $\bar{I}(e_n)=\infty$ for some n, we are done. If not, given $\varepsilon>0$, there is an element $j_n \in U$ such that $e_n=j_n$ and $I(j_n) \leq \bar{I}(e_n)+\varepsilon \cdot 2^{-n}$. Since each j_n is in P, Lemma 2.1 implies that the element $j=\sum j_n$ is in U and that $I(j)=\sum I(j_n) \leq \sum \bar{I}(e_n)+\varepsilon$. Since $j \geq e$, we have $\bar{I}(e) \leq \sum_{n=1}^{\infty} \bar{I}(e_n)+\varepsilon$, and the lemma follows since ε was an arbitrary positive number.

PROPOSITION 2.3. *Let $\{e_n\}$ be an increasing sequence of elements in $L^1(I)$, and let $e=\lim e_n$. Then $e \in L^1(I)$ if and only if $\lim I(e_n)<\infty$. In this case*

$I(e)=\lim I(e_n)$.

PROOF. Since $e\geq e_n$, $\bar{I}(e)\geq I(e_n)$. Thus if $\lim I(e_n)=\infty$, then $\bar{I}(e)=\infty$, and $e\notin L^1(I)$. Suppose $\lim I(e_n)<\infty$. Set $j=e-e_1$. Then $j\geq 0$, and $j=\sum_{n=1}^{\infty}(e_{n+1}-e_n)$. Hence by Lemma 2.2, $\bar{I}(j)\leq\sum_{n=1}^{\infty} I(e_{n+1}-e_n)=\sum_{n=1}^{\infty} I(e_{n+1})-I(e_n)=\lim I(e_n)-I(e_1)$ Thus $\bar{I}(e)=\bar{I}(e_1+j)\leq I(e_1)+\bar{I}(j)\leq\lim I(e_n)$. Since $e_n\leq e$, we have $\underline{I}(e)\geq I(e_n)$, and so $\underline{I}(e)\geq\lim I(e_n)$. Thus $\underline{I}(e)=\bar{I}(e)=I(e)=\lim I(e_n)$.

THEOREM 2.4. (Fatou's Lemma). *Let $\{e_n\}$ be a sequence in $P\cap L^1(I)$. Then the element $\inf e_n$ is in $L^1(I)$, and the element $\varliminf e_n$ is in $L^1(I)$ if $\varliminf I(e_n)<\infty$. In this case $I(\varliminf e_n)\leq\varliminf I(e_n)$.*

PROOF. Let $j_n=e_1\wedge e_2\wedge\cdots\cdots\wedge e_n$. Then $\{j_n\}$ is a sequence in $P\cap L^1(I)$, which decrease to $j=\inf e_n$. Thus $-j_n\uparrow -j$, and since $I(-j_n)=-I(j_n)\leq 0$, we must have $j\in L^1(I)$ by Proposition 2.3. To prove the rest of the theorem, let $k_n=\inf\{e_k|k\geq n\}$. Then $\{k_n\}$ is a sequence in $P\cap L^1(I)$ which increase to $\varliminf e_n$. Since $j_n\leq e_n$ for $n\leq k$, $\lim I(j_n)\leq\varliminf I(e_k)<\infty$. Hence $\varliminf e_k\in L^1(I)$ and $I(\varliminf e_n)\leq\varliminf I(e_n)$ by Proposition 2.3.

PROPOSITION 2.5. *Let $\{e_n\}$ be a sequence of elements in $L^1(I)$ and suppose that there is an element j in $L^1(I)$ such that for all n we have $|e_n|\leq j$. Then if $e=\lim e_n$, we have $I(e)=\lim I(e_n)$.*

PROOF. The elements e_n+j are in P, and $I(e_n+j)\leq 2I(j)$. Hence by Theorem 2.4, we have $e+j$ in $L^1(I)$ and $I(e+j)\leq\varliminf I(e_n+j)=I(j)+\varliminf I(e_n)$. Hence $I(e)\leq\varliminf I(e_n)$. Since the element $j-e_n$ are also in P, we have $I(j-e)=\varliminf(j-e_n)$ $=I(j)-\varlimsup I(e_n)$. Hence $\varlimsup I(e_n)\leq I(e)$, and so $\lim I(e_n)$ exists and is equal to $I(e)$.

§ 3. Extension of I.

It follows from Proposition 2.3 applied to $\{-e_n\}$ that

LEMMA 3.1. *If e is any element with $I(e)$ finite, then there is an element j in $U\cup -U$ such that $e\leq j$ and $\bar{I}(e)=I(j)$.*

PROOF. Let e be any element with $I(e)$ finite. Then given n we can find $j_n\in U$ such that $e=j_n$ and $I(j_n)\leq\bar{I}(e)+\frac{1}{n}$. Setting $k_n=j_1\wedge j_2\wedge\cdots\wedge j_n$, we have $e\leq k_n\leq j_n$, and so $\{k_n\}$ is a decreasing sequence of elements in U with $I(e)\leq$

$I(k_n) \leq \bar{I}(e) + \frac{1}{n}$. Hence the element $k = \lim k_n$ is in $U \cup -U$ while $e \leq k$ and $\bar{I}(e) = I(k)$. We have thus established the lemma.

DEFINITION 3.2. An element e is called *null* if $e \in L^1(I)$ and $I(|e|) = 0$.

If e is a null element and $|j| \leq e$, $0 \leq \underline{I}(|j|) \leq \bar{I}(|j|) \leq I(e) = 0$. Hence $j \in L^1(I)$, and j is a null element.

PROPOSITION 3.3. *An element e is in $L^1(I)$ if and only if e is the difference j-k of an element j in $U \cup -U$ and a null element k in P. An element k is a null element if and only if there is a null element l in $U \cup -U$ such that $|j| \leq l$.*

PROOF. If $e = j - k$, then e is the difference of two elements in $L^1(I)$ and so must itself be in $L^1(I)$. If $|j| \leq l$ with l null, then j is a null element. If e is in $L^1(I)$, then Lemma 3.1. asserts the existence of j in $U \cup -U$ such that $e \leq j$ and $I(e) = I(j)$. Hence $k \geq j - e$ is an element in P and $I(k) = 0$, making k a null elemant. If k is a null element, then by Lemma 3.1. there is an element l in $U \cup -U$ with $|j| \leq l$ and $I(l) = I(|k|) = 0$.

PROPOSITION 3.4. *Let I be an elementary integral on a vector lattice L and let J be an elementary integral on a vector lattice $D \supset L$. If $I(e) = J(e)$ for all $e \in L$, then $D^1(I) \supset L^1(I)$ and $I(e) = J(e)$ for all $e \in L^1(I)$.*

PROOF. By applying Proposition 2.3. twice, we see that $U \cup -U \subset D^1(I)$ and that $I(e) = J(e)$ for e in $U \cup -U$. Hence by the second part of Proposition 3.3., each element which is null with respect to I must also be null with respect to J. By the first part of Proposition 3.3., every element e in $L^1(I)$ must be in $D^1(I)$, and $I(e) = J(e)$.

§4. I and L-measurability on $L^1(I)$.

We now turn our attention to the I and L-measurability on the I-summable class $L^1(I)$.

DEFINITION 4.1. For an element e in E, e is said to be *I-measurable* if $e \wedge a$ is in $L^1(I)$ for each a in L.

DEFINITION 4.2. For an element e in E, e is said to be *L-measurable* if $\sup\{I(a) \mid a \leq e,\ a \in -U\} = \inf\{I(b) \mid b \geq e,\ b \in U\}$.

THEOREM 4.3. *If an element e in E is I-measurable, then it is L-measurable.*

PROOF. Since we always have

$$\sup\{I(a)\mid a\leq e,\ a\in -U\}\leq I(e)\leq \inf\{I(b)\mid b\geq e,\ b\in U\},$$

we see that e is L-measurable if

$$\sup\{I(a)\mid a\leq e,\ a\in -U\}\geq \inf\{I(b)\mid b\geq e,\ b\in U\}.$$

We assume that for each a in L, $a\wedge e$ in $L^1(I)$, i.e. for arbitrary $\varepsilon>0$, there exist $y\in -U$ and $z\in U$ such that $y\leq e\wedge a\leq z$ with $0\leq I(z)-I(y)<\varepsilon$.

There may be two cases for any element e.

Case 1: $y\leq e\wedge a\leq e\leq z$. In this case $y\leq e\leq z$, we have $0\leq I(z)-I(y)<\varepsilon$. Hence $I(z)<I(y)+\varepsilon$, and so $\inf\{I(s)\mid s\geq e,\ s\in U\}\leq I(z)<I(y)+\varepsilon$. From the monotonity of I, $\inf\{I(s)\mid s\geq e,\ s\in U\}-I(y)\geq 0$. This means that $0\leq \inf\{I(s)\mid s\geq e,\ s\in U\}-I(y)<\varepsilon$. Analogously $\inf\{I(s)\mid s\geq e,\ s\in U\}<\sup\{I(y)\mid y\leq e,\ y\in -U\}+\varepsilon$. Hence $\inf\{I(s)\mid s\geq e,\ s\in U\}\leq \sup\{I(y)\mid y\leq e,\ y\in -U\}$.

Case 2: $y\leq e\wedge a\leq z<e$. Since for any a in L, $e\wedge a\in L^1(I)$ the element e itself must be in $L^1(I)$. Hence there exist s in $-U$, and t in U such that $s\leq e\leq t$ with $0\leq I(t)-I(s)<\varepsilon$. i.e. $I(t)<I(s)+\varepsilon$ where $t\in U$, $e\leq t$. Therefore $\inf\{I(l)\mid e\leq l,\ l\in U\}\leq I(t)<I(s)+\varepsilon$, and $0\leq \inf\{I(t)\mid e\leq t,\ t\in U\}-I(s)<\varepsilon$. Analogously $\inf\{I(t)\mid e\leq t,\ t\in U\}<\sup\{I(s)\mid s\leq e,\ s\in U\}+\varepsilon$. Hence $\inf\{I(t)\mid e\leq t,\ t\in U\}\leq \sup\{I(s)\mid s\leq e,\ s\in -U\}$.

THEOREM 4.4. *If I is an infinitely $\wedge$-distributive, $\wedge$-associative lattice homomorphism, and e is L-measurable, then it is I-measurable.*

PROOF. Assume that $\sup\{I(a)\mid a\leq e,\ a\in -U\}=\inf\{I(b)\mid b\geq e,\ b\in U\}$. We have
$\sup\{I(a\wedge x)\mid a\wedge x\leq e\wedge x,\ a\wedge x\in -U\}=\inf\{I(b\wedge x)\mid b\wedge x\geq e\wedge x,\ b\wedge x\in U\}$

for an arbitrary (but fixed) element x in L.

If $I(x)=0$, then $x=0$ and $a\wedge x=0$ for $a\geq 0$, $e\wedge x=a$ for $a<0$. Hence $a\wedge x$ is in $L^1(I)$. Suppose $I(x)\neq 0$. From the above equation we have

$$\sup\{I(a)\wedge I(x)\mid a\wedge x\leq e\wedge x,\ a\wedge x\in -U\}=\inf\{I(b)\wedge I(x)\mid b\wedge x\geq e\wedge x,\ b\wedge x\in U\}.$$

Since $I(x)$ is a constant real number,

$$I(x)\wedge\sup\{I(a)\mid a\wedge x\leq e\wedge x,\ a\wedge x\in -U\}=I(x)\wedge\inf\{I(b)\mid b\wedge x\geq e\wedge x,\ b\wedge x\in U\},$$

and so $\sup\{I(a)\mid a\wedge x\leq e\wedge x,\ a\wedge x\in -U\}=\inf\{I(b)\mid b\wedge x\geq e\wedge x,\ b\wedge x\in U\}$.

By putting $y=a\wedge x$, $z=b\wedge x$, it is easy to see that $y\in -U$ and $z\in U$. Thus we have $\sup\{I(a)\mid y\leq e\wedge x,\ a\wedge x\in -U\}=\inf\{I(b)\mid z\geq e\wedge x,\ b\wedge x\in U\}$, where $e\wedge x$ is in $L^1(I)$ for any element x in L. Since $a\wedge x\leq x\wedge x\leq b\wedge x$ with

$I(b \wedge x) - I(a \wedge x) < \varepsilon$, by the assumption, e is I-measurable.

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REFERENCES

[1] Howard Anton and W. J. Pervin. *Integration on topological semifields,* Pacific Journal of Math., 1971.

[2] H. L. Royden. Real Analysis, Macmillan, New York. 1971.

[3] Anthony L. Peressini. Ordered Topological Vector Spaces, Harper and Row, New York, 1967.

[4] G. Birkhoff. Lattice Theory, American Mathematical Society, 1948.

[5] G. Choquet. Lectures on Analysis, Vol. 1, W. A. Benjamin, Inc, 1969.