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A RELATION SATISFIED BY SOLUTIONS OF THE ADJOINT EQUATION

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Let $y_1, y_2, \cdots, y_n$ be n linearly independent solutions of the differential equation

$$y^{(n)}+p_{n-1}y^{(n-1)}+\cdots+p_0y=0, \tag{1}$$

where $p_i \in C^i$, $i=0,1,\cdots, n-1$, and let $W=|y_j^{(i-1)}|_{i,j=1}^n$ be the Wronskian. It is well-known [1] that

$$v=\frac{1}{W}\begin{vmatrix} y_1 & y_2 & \cdots & y_{n-1} \\ y_1' & y_2' & \cdots & y_{n-1}' \\ & \cdots\cdots & & \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_{n-1}^{(n-2)} \end{vmatrix} \tag{2}$$

is a solution of the adjoint equation

$$v^{(n)}-(p_{n-1}v)^{(n-1)}+(p_{n-2}v)^{(n-2)}-\cdots+(-1)^n p_0 v=0. \tag{3}$$

We shall prove the following generalization of (2).

THEOREM 1. *If $y_1, y_2, \cdots, y_n$ are n linearly independent solutions of* (1), *there exist n linearly independent solutions $v_1, v_2, \cdots, v_n$ of* (3) *such that*

$$\begin{vmatrix} v_1 & v_2 & \cdots & v_k \\ v_1' & v_2' & \cdots & v_k' \\ & \cdots\cdots & & \\ v_1^{(k-1)} & v_2^{(k-1)} & \cdots & v_k^{(k-1)} \end{vmatrix} = \frac{1}{W}\begin{vmatrix} y_1 & y_2 & \cdots & y_{n-k} \\ y_1' & y_2' & \cdots & y_{n-k}' \\ & \cdots\cdots & & \\ y_1^{(n-k-1)} & y_2^{(n-k-1)} & \cdots & y_{n-k}^{(n-k-1)} \end{vmatrix} \tag{4}$$

$$k=1,2,\cdots, n-1.$$

For the proof of this theorem, we require a few results from the theory of determinants. Each element a_{ij} of the determinant $D=|a_{ij}|_{i,j=1}^n$ has a cofactor A_{ij}. Put $\Delta=|A_{ij}|_{i,j=1}^n$. Then it is easily confirmed that $D\Delta=D^n$. If $D\neq 0$, we have

$$\Delta=D^{n-1}. \tag{5}$$

If $(n-m)$ rows and $(n-m)$ columns in D are deleted, there results an $m\times m$

determinant $M=|a_{r_i s_j}|_{i,j=1}^{m}$. This determinant M is called an *m th-order minor* of D. On the other hand, if we delete from D the rows and columns to which the elements of M belong, we get an $(n-m)\times(n-m)$ determinant N. N is called the *complement* of M. The *algebraic complement* $\overline{M}$ of an mth-order minor M is defined to be $(-1)^{r_1+\cdots+r_m+s_1+\cdots+s_m}N$.

LEMMA 1 [3]. *Let $\mathfrak{M}$ be a pth-order minor of Δ, and M the corresponding minor of D (i.e., M has the same row and column indices as $\mathfrak{M}$). Then*

$$\mathfrak{M}=D^{p-1}\overline{M},$$

provided $D\neq 0$.

Let $\mathfrak{S}_n$ be the set of all permutations of the integers between 1 and n. Then

$$D=\sum_{\pi\in\mathfrak{S}_n}(\operatorname{sgn}\pi)a_{1\pi(1)}a_{2\pi(2)}\cdots a_{n\pi(n)},$$

where $\operatorname{sgn}\pi$ is $+1$ or -1 according as π is even or odd. From this representation of D, we easily deduce the following lemma.

LEMMA 2.

$$D=|a_{ij}|_{i,j=1}^{n}=|(-1)^{i+j}a_{ij}|_{i,j=1}^{n}=|a_{n+1-i,n+1-j}|_{i,j=1}^{n}.$$

We are now ready for the proof of Theorem 1.

Proof of Theorem 1. Let σ_{ij} be the minor of $y_j^{(i-1)}$ in W. Then $v_k=\sigma_{n,n+1-k}/W$, $k=1,2,\cdots,n$, are solutions of (3) [1]. To prove the linear independence, it suffices to show that the Wronskian $\mathfrak{W}=|v_j^{(i-1)}|_{i,j=1}^{n}$ does not vanish. Put $\mathfrak{B}_k=|v_j^{(i-1)}|_{i,j=1}^{k}$ and $\Sigma_k=|\sigma_{ij}|_{i,j=k}^{n}$. Then it is easily confirmed that

$$\mathfrak{B}_k=\frac{1}{W^k}\Sigma_{n+1-k},\quad k=1,\ 2,\ \cdots,\ n. \tag{6}$$

Hence,

$$\mathfrak{W}=\mathfrak{B}_n=\frac{1}{W^n}\Sigma_1=\frac{1}{W^n}W^{n-1}=\frac{1}{W},$$

the third equality following from (5) and Lemma 2. Therefore, the Wronskian $\mathfrak{W}$ does not vanish.

By Lemmas 1 and 2, we have

$$\Sigma_{n+1-k}=W^{k-1}|y_j^{(i-1)}|_{i,j=1}^{n-k}, \tag{7}$$

$k=1,2,\cdots,n-1$. From (6) and (7), we conclude

$$\mathfrak{B}_k=\frac{1}{W}|y_j^{(i-1)}|_{i,j=1}^{n-k},$$

$k=1, 2, \cdots n-1$, establishing (4).

We remark that (4) holds for $k=n$ if we set $|y_j^{(i-1)}|_{i,j=1}^{0}=1$.

A solution y of (1) is said to have a *zero of order* k at ξ if $y(\xi)=y'(\xi)=\cdots=y^{(k-1)}(\xi)=0$: if further $y^{(k)}(\xi)\neq 0$. we say that y has a *zero of order exactly* k at ξ.

THEOREM 2. *If* (1) *has a nontrivial solution* y *with a zero of order* k *at* ξ *and a zero of order* n-k *at* ζ, *then* (3) *has a nontrivial solution* v *with a zero of order* n-k *at* ξ *and a zero of order* k *at* ζ.

PROOF. Let $y_1, y_2, \cdots, y_n$ be solutions of (1) satisfying $y_j^{(n-i)}(\xi)=\delta_{ij}, i, j=1, 2, \cdots, n$. Then the v_k, as defined in Theorem 1, satisfies

$$v_k(\xi)=v'_k(\xi)=\cdots=v_k^{(n-k-1)}(\xi)=0, \tag{8}$$

$k=1, 2, \cdots n-1$. Since the y has a zero of order k at ξ, $y=c_1y_1+c_2y_2+\cdots+c_{n-k}y_{n-k}$ for some constants $c_1, c_2, \cdots, c_{n-k}$, not all zero. Furthermore, $|y_j^{(i-1)}(\zeta)|_{i,j=1}^{n-k}=0$ because the y has a zero of order $n-k$ at ζ. In view of (4), this implies $|v_j^{(i-1)}(\zeta)|_{i,j=1}^{k}=0$. Hence, there exists a set of constants $C_1, C_2, \cdots, C_k$ such that $v=C_1v_1+\cdots+C_kv_k$ is a nontrivial solution of (3) with a zero of order k at ζ. That this v has a zero of order $n-k$ at ξ is immediate from (8).

Sherman [5, Theorem 10] obtained a similar result under the stronger condition that y have a zero of order exactly k at ξ and a zero of order $n-k$ at ζ.

The equation

$$(3\sin^2 x\cos^2 x-2)y'''-6\sin x\cos x(\cos^2 x-\sin^2 x)y''-(9\sin^2 x\cos^2 x+14)y'=0, \tag{9}$$

where $3\sin^2 x\cos^2 x-2<0$, has three linearly independent solutions $\sin^2 x\cos x$, $\cos^2 x\sin x$, and 1 [6]. The solution $\cos^2 x\sin x$ has double zeros at $-\pi/2$ and $\pi/2$. Therefore, (9) cannot have a solution with a zero of order exactly 1 at $-\pi/2$ and a zero of order 2 at $\pi/2$. This example shows that the condition in [5, Theorem 10] is indeed stronger than that in Theorem 2.

Suppose $p_0, p_1, \cdots, p_{n-1}$ in (1) are real-valued, continuous functions defined on an interval I. For the even-order equation $(n=2m)$, we say that (1) is *disconjugate in the sense of Reid* [4] on I if none of its nontrivial solutions have two mth-order zeros on I. By Theorem 2 we see that (1) is disconjugate in the sense of Reid if and only if (3) is disconjugate in the sense of Reid. Moreover, if $P_i \in C^i(I)$, (3) may be cast into the form of (1):

$$v^{(n)}+q_{n-1}v^{(n-1)}+\cdots+q_0v=0,$$

where

$$q_i=\sum_{k=i}^{n-1}(-1)^{n-k}\binom{k}{i}p_k^{(k-i)}, \tag{10}$$

$i=0, 1, \cdots, n-1$. Since (1) with $n=2m$ is known to be disconjugate in the sense of Reid on $(-c, c)$ if

$$\sum_{k=1}^{m}\frac{|P_{2m-k}(x)|}{k!}(c+|x|)^k+\sum_{k=m+1}^{2m}\frac{|p_{2m-k}(x)|}{k!}(c-|x|)^{k-m}(c+|x|)^m\leq 1$$

[2, Theorem 2.3], we have the following result.

THEOREM 3. *Assume that* $p_i \in C^i, i=0, 1, \cdots, 2m-1$, *is a real-valued function defined on* $(-c, c)$. *The differential equation*

$$y^{(2m)}+p_{2m-1}y^{(2m-1)}+\cdots+p_0y=0$$

is disconjugate in the sense of Reid on $(-c, c)$ *if*

$$\sum_{k=1}^{m}\frac{|q_{2m-k}(x)|}{k!}(c+|x|)^k+\sum_{k=m+1}^{2m}\frac{|q_{2m-k}(x)|}{k!}(c-|x|)^{k-m}(c+|x|)^m\leq 1,$$

where $q_0, q_1, \cdots, q_{2m-1}$ *are defined as in* (10) *with* $n=2m$.

By using Theorem 3, the differential equation

$$y^{(2m)}+\left[\frac{(2m-1)!}{2(1-x^2)^{m-1}}y\right]'=0 \tag{11}$$

is easily shown to be disconjugate in the sense of Reid on $(-1, 1)$. However, Theorem 2. 3 in [2] is inconclusive as to the disconjugacy of (11).

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