Kyungpook Math. J.
Volume 13, Number 1
June, 1973

ON CONDUCTION OF HEAT IN A SEMI INFINITE CIRCULAR CYLINDER

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1. The object of this paper is to employ the Appell function F_4 [1, p.14] to obtain the solution of the differential equation of the conduction of heat in a semi-infinite circular cylinder.

Below we obtain some results required in the proof:

$$\int_0^\infty e^{-kt\xi^2}\,{}_1F_2\left[\begin{matrix}1, \\ 3/2,\ m+2,\end{matrix}\ -\frac{b^2\xi^2}{4}\right](\xi)\sin(z\xi)\,d\xi$$

$$=\frac{\sqrt{\pi}(z/4)}{(kt)^{3/2}}F^{1,1}_{0,2}\left[\begin{matrix}3/2:1; & 1; \\ :3/2,\ m+2:3/2,\ 1;\end{matrix}\left|\begin{matrix}-b^2/4kt \\ -z^2/4kt\end{matrix}\right.\right] \tag{1.1}$$

$$\int_0^a (r)^{\rho-1}\left(1-\frac{r}{a}\right)^{\nu-1}{}_2F_1\left[\begin{matrix}-n,\ \beta+n, \\ \nu,\end{matrix}\ 1-\frac{r}{a}\right]J_0(\eta_i r)\,dr$$

$$=\frac{\Gamma(\nu)\ \Gamma(\rho)\ \Gamma(\nu+\rho-\beta)\ (a)^\rho}{\Gamma(\nu+\rho+n)\ \Gamma(\nu+\rho-\beta-n)}$$

$$\times{}_4F_5\left[\begin{matrix}\Delta(2,\rho),\ \Delta(2,\nu+\rho-\beta); \\ 1,\ \Delta(2,\nu+\rho+n),\ \Delta(2,\nu+\rho-\beta-n);\end{matrix}\ -1/4a^2\eta_i^2\right] \tag{1.2}$$

provided $\mathrm{Re}(\nu)>0$, $\mathrm{Re}(\rho)>0$, $\mathrm{Re}(\nu+\rho-\beta)>0$, *and* $\Delta(m,\alpha)$ *stands for* $\frac{\alpha}{m}$, $\frac{\alpha+1}{m}$, $\dots$, $\frac{\alpha+m-1}{m}$.

$$\int_0^\infty e^{-u^2/4kt}\,{}_2F_1\left[\begin{matrix}-n,\ \beta+n, \\ \beta-\nu+1,\end{matrix}\ b^2u^2\right]\sinh\left(\frac{uz}{2kt}\right)du$$

$$=(z)\sum_{s=0}^{\infty}\frac{(1)_s}{(3/2)_s}(z^2/4kt)^s\,{}_3F_1\left[\begin{matrix}-n,\ \beta+n,\ s+1 \\ \beta-\nu+1,\end{matrix};\ 4b^2kt\right] \tag{1.3}$$

In (1.2) *and* (1.3), n *is a positive integer.*

PROOF. i) Expressing the hypergeometric function ${}_1F_2$ in series form, and evaluating the integral by [2, p. 74(24)], we have,

$$\int_0^\infty e^{-kt\xi^2}\,{}_1F_2\left[\begin{matrix}1, & & -\dfrac{b^2\xi^2}{4}\\ 3/2, & m+2, & \end{matrix}\right](\xi)\sin(z\xi)\,d\xi$$

$$=\frac{\sqrt{\pi}(kt)^{-3/2}\,(z)}{4}\sum_{s=0}^{\infty}\frac{(3/2)_s(1)_s(-b^2/4kt)^s}{s!\,(3/2)_s(m+2)_s}\,{}_1F_1\left[\begin{matrix}s+3/2, & -z^2/4kt\\ 3/2, & \end{matrix}\right]$$

Now writing the function ${}_1F_1$ in series form and taking the sum of both the series on right hand side, we obtain the required result (1.1) where $F^{1,1}_{0,2}$ stands for the Kampe de Feriet function, [1, p. 150].

ii) In left hand side of (1.2), replacing $1-\frac{r}{a}$ by t, expressing the Bessel function as a series, evaluating the integral by [3, p. 399(4)] and simplifying, we obtain the result (1.2).

iii) In left hand side of (1, 3), expressing the hyperbolic function $\sinh(uz/2kt)$ as a series, evaluating the integral by [4, p. 48, ex-17, (iii)] and simplifying we obtain the result (1.3).

2. In [5, pp. 181-183], the problem of determining the distribution of temperature in a semi-infinite circular cylinder is considered. The cylinder is of radius 'a' whose faces $z=0$ and $r=a$ are kept at zero temperature and the flow of heat results from the initial distribution of temperature throughout the cylinder. The temperature $\theta(r, z, t)$ must satisfy the partial differential equation

$$\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\,\frac{\partial\theta}{\partial r}+\frac{\partial^2\theta}{\partial z^2}=\frac{1}{k}\,\frac{\partial\theta}{\partial t}$$

subject to the boundary and initial conditions as

$$\theta=0,\ z=0,\ 0\le r\le a,\ t>0.$$

$$\theta=0,\ r=a,\ z>0,\ t>0,$$

$$\theta(r, z, 0)=f(r, z).$$

The solution of the above boundary value problem as given by Snedden [5, p. 182] is

$$\theta=2/a^2\ (2/\pi)^{1/2}\sum_i\frac{J_0(\eta_i r)\ e^{-kt\eta_i^2}}{J_1^2(\eta_i a)}\int_0^\infty F(\eta_i\xi)\ e^{-kt\xi^2}\sin(z\xi)\,d\xi \quad (2.1)$$

where the sum is taken over all the positive roots of the equation $J_0(\eta_i a)=0$ and

$$F(\eta_i\xi)=\sqrt{\frac{2}{\pi}}\int_0^\infty\sin(\xi z)dz\int_0^a rf(r, z)\ J_0(\eta_i r)dr \quad (2.2)$$

Particular cases: i) If $f(r, z)=f(r)\ g(z)$ then

$$\theta=\frac{2e^{-z^2/4kt}}{a^2\sqrt{\pi kt}}\sum_i \frac{J_0(\eta_i r)\bar{f}_J(\eta_i)\,e^{-kt\eta_i^2}}{J_1^2(\eta_i a)}$$

$$\times\int_0^\infty g(u)\sinh(uz/2kt)\,e^{-u^2/4kt}\,du \tag{2.3}$$

$$\text{where } \bar{f}_J(\eta_i)=\int_0^a r\,f(r)\,J_0(\eta_i r)\,dr \tag{2.4}$$

ii) If $f(r,z)=g(z)$;

then the solution reduces to

$$\theta=\frac{2\,e^{-z^2/4kt}}{\sqrt{\pi kt}}\sum_i \frac{J_0(\eta_i r)\,e^{-kt\eta_i^2}}{(a\eta_i)\,J_1(a\eta_i)}$$

$$\times\int_0^\infty g(u)\sinh(uz/2kt)\,e^{-u^2/4kt}\,du \tag{2.5}$$

3. In this article, we have employed the Appell function F_4 to obtain the solution of the differential equation of the conduction of heat in a semi-infinite circular cylinder considered in the last article.

i) Let

$$f(r,z)=(r)^{\rho-2}(a-r)^{\nu-1}F_4\left[\alpha,\beta;\nu,\delta;(a-r)(b^2-z^2),(z^2-b^2)r\right]$$

substituting this value of $f(r,z)$ in (2.2), we have after simplification,

$$F(\eta_i\xi)=\left(\frac{2}{\pi}\right)^{1/2}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_{m+n}(-1)^n}{m!n!(\nu)_m(\delta)_n}$$

$$\times\int_0^\infty(b^2-z^2)^{m+n}\sin(\xi z)dz\int_0^a(r)^{\rho+n-1}(a-r)^{\nu+m-1}J_0(\eta_i r)\,dr$$

Evaluating the integrals by [3, p. 193, (56)] and [2, p. 69, (7) and p. 372], we have

$$F(\eta_i\xi)=\frac{B(\rho,\nu)\,(\xi)(a)^{\rho+\nu-1}}{(2\pi)^{1/2}}$$

$$\times\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_{m+n}(1)_{m+n}(\rho)_n(-1)^n(b)^{2m+2n+2}(a)^{m+n}}{m!\,n!\,(2)_{m+n}(\delta)_n(\rho+\nu)_{m+n}}$$

$$\times\,{}_1F_2\left[\begin{matrix}1, \\ 3/2,\ m+n+2,\end{matrix}\ -b^2\xi^2/4\right]{}_2F_3\left[\begin{matrix}\Delta(2,\rho+n), \\ 1,\ \Delta(2,\rho+\nu+m+n),\end{matrix}\ -\eta_i^2a^2/4\right]$$

on taking the Cauchy's product between the series in m and n, we obtain after simplification,

$$F(\eta_i\xi)=\frac{B(\rho,\nu),\ (\xi)(a)^{\rho+\nu-1}}{(2\pi)^{1/2}}$$

$$\times\sum_{m,r=0}^{\infty}\frac{(\alpha)_m\,(\beta)_m\,(\rho)_{2r}(a)^m\,(b)^{2m+2}\left(\frac{-\eta_i^2a^2}{4}\right)^r}{(2)_m\,(\rho+\nu)_{m+2r}\,r!\,(1)_r}$$

$$\times{}_1F_2\left[\begin{matrix}1, & \\ \frac{3}{2}, & m+2,\end{matrix}\ \frac{-b^2\xi^2}{4}\right]{}_2F_1\left[\begin{matrix}-m, & \rho+2r, \\ & \delta\end{matrix}\ 1\right].$$

Making use of the results [4, p. 23, (7.2)] and

$\Gamma(a-n)=\frac{\Gamma(a)\ (-1)^n}{(1-a)_n}$, we have on summing the series of r,

$$F(\eta_i\xi)=\frac{B(\rho,\nu)\ (\xi)\ (a)^{\rho+\nu-1}}{(2\pi)^{1/2}}\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m\,(\delta-\rho)_m\,(a)^m\,(b)^{2m+2}}{(2)_m\,(\rho+\nu)_m\,(\delta)_m}$$

$$\times{}_1F_2\left[\begin{matrix}1, & \\ 3/2, & m+2,\end{matrix}\ -b^2\xi^2/4\right]\ {}_4F_5\left[\begin{matrix}\Delta(2,\rho),\Delta(2,\rho-\delta+1), \\ 1,\ \Delta(2,\rho+\nu+m),\ \Delta(2,\rho+\delta-m+1)\end{matrix}\ \frac{-\eta_i^2a^2}{4}\right]$$

Substituting this value of $F(\eta_i\xi)$ in (2.1) and evaluating the integral with the help of (1.1) we obtain finally,

$$\theta(r,z,t)=\frac{B(\rho,\nu),\ (z/2)\ (a)^{\rho+\nu-3}}{\sqrt{\pi}\ (kt)^{3/2}}\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m(\delta-\rho)_m\ (a)^m\ (b)^{2m+2}}{(2)_m\ (\rho+\nu)_m(\delta)_m}$$

$$\times F^{1,1}_{0,2}\left[\begin{matrix}3/2:1; & 1: \\ \text{——}:3/2, & m+2;3/2,1;\end{matrix}\ \middle|\ \begin{matrix}-b^2/4kt \\ -z^2/4kt\end{matrix}\right]$$

$$\times\sum_i\frac{J_0(\eta_i r).e^{-kt\eta_i^2}}{J_1^2(\eta_i a)}{}_4F_5\left[\begin{matrix}\Delta(2,\rho),\ \Delta(2,\rho-\delta+1), & -\eta_i^2a^2/4 \\ 1,\ \Delta(2,\rho+\nu+m),\ \Delta(2,\rho-\delta-m+1), & \end{matrix}\right]$$

ii) Let

$$f(r,z)=(r)^{\rho-2}\left(1-\frac{r}{a}\right)^{\nu-1}F_4\left[-n,\ \beta+n;\nu,\ \beta-\nu+1;\left(1-\frac{r}{a}\right)(1-b^2z^2),\ \frac{b^2z^2r}{a}\right].$$

Making use of the formula [6, p. 82, (1)], (3.2) can be written as

$$f(r,z)=f(r)\ g(z)$$

where $f(r)=(r)^{\rho-2}\left(1-\frac{r}{a}\right)^{\nu-1} {}_2F_1\left[\begin{matrix}-n, \beta+n, \\ \nu\end{matrix}\; 1-\frac{r}{a}\right]$

and $g(z)={}_2F_1\left[\begin{matrix}-n, \beta+n, \\ \beta-\nu+1,\end{matrix}\; b^2z^2\right]$

Substituting these values of $f(r)$ and $g(z)$ in (2.3) and evaluating the integrals by (1.2) and (1.3), we have on obvious simplification,

$$\theta=\frac{\Gamma(\nu)\ \Gamma(\rho)\ \Gamma(\nu+\rho-\beta)\ (2z)\ (a)^{\rho-2}\ e^{-z^2/4kt}}{\Gamma(\nu+\rho+n)\ \Gamma(\nu+\rho-\beta-n)\ (\pi kt)^{1/2}}$$

$$\times\sum_{s=0}^{\infty}\frac{(z^2/4kt)^s}{(3/2)_s}\ {}_3F_1\left[\begin{matrix}-n, \beta+n, 1+s, \\ \beta-\nu+1,\end{matrix}\; 4b^2kt\right]\sum_i\frac{J_0(\eta_i r)\ e^{-kt\eta_i^2}}{J_1^2(\eta_i a)}$$

$$\times{}_4F_5\left[\begin{matrix}\Delta(2, \rho), \Delta(2, \nu+\rho-\beta), \\ 1, \Delta(2, \nu+\rho+n), \Delta(2, \nu+\rho-\beta-n),\end{matrix}\; -\frac{1}{4}a^2\eta_i^2\right]$$

iii) If $f(r)=1$ and $g(z)={}_2F_1\left[\begin{matrix}-n, \beta+n, \\ \beta-\nu+1,\end{matrix}\; b^2z^2\right]$ (3.3)

then substituting this value of $g(z)$ in (2.5) and evaluating the integral by (1.3), we have the result

$$\theta(r, z, t)=\frac{(2z)\ e^{-z^2/4kt}}{(\pi kt)^{1/2}}\sum_{s=0}^{\infty}\frac{(z^2/4kt)^s}{(3/2)_s}\ {}_3F_1\left[\begin{matrix}-n, \beta+n, 1+s, \\ \beta-\nu+1,\end{matrix}\; 4b^2kt\right]$$

$$\times\sum_i\frac{J_0(\eta_i r)\ e^{-kt\eta_i^2}}{(a\eta_i)J_1(a\eta_i)} \qquad (3.1)$$

The hypergeometric polynomials $f(r)$ and $g(z)$ used in obtaining the results (3.3) and (3.4) can be reduced to known polynomials just as Jacobi, Laguerre etc. and the special cases can be obtained. Similarly the Appell function F_4 employed in obtaining (3.1) can be reduced to confluent hypergeometric function ϕ_2 [1, p. 126, 5°] which in turn can be reduced to Whittakar function of two variables [1, p. 132, (29)] and interesting results can be obtained.

Acknowledgement

I take this opportunity of expressing my gratitude to Dr. V. M. Bhise of G. S. Institute of Technology and Science, Indore, for his valuable guidance in the preparation of this paper. My thanks are also due to Dr. S. M. Das Gupta, Director, G. S. I. T. S., Indore for the facilities he provided me.

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