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ON THE MEANS OF THE PRODUCT OF TWO ENTIRE FUNCTIONS

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1. Introduction. Let $f(z)$ and $g(z)$ be two entire functions of orders ρ_f, ρ_g and lower orders λ_f, λ_g. Let us define the means:

$$(1.1)\qquad I_{\alpha,\beta}=\left\{\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^{\alpha}|g(re^{i\theta})|^{\beta}d\theta\right\}^{\frac{1}{\alpha+\beta}},$$

and

$$(1.2)\qquad I^{(k)}_{\alpha,\beta}=\frac{1}{r^{k+1}}\int_0^r x^k I_{\alpha,\beta}(x)dx,$$

where $\alpha>0$, $\beta>0$ and $k+1>0$.

Extending to two functions the results proved earlier by Rahman [6], [7], Lakashminarasimhan [4] has proved:

$$(1.3)\qquad (\alpha+\beta)I_{\alpha,\beta}(r)\sim\log\left[\max_{|z|=r}|f^{\alpha}g^{\beta}|\right],\ r\to\infty,$$

and

$$(1.4)\qquad \overline{\lim_{r\to\infty}}\left\{\frac{I_{\alpha,\beta}(r)}{I^{(k)}_{\alpha,\beta}(r)}\right\}^{\frac{1}{\log r}}\leq e^{\rho_f+\rho_g},\ \text{provided } \rho_f,\ \rho_g<\infty.$$

My chief aim in this note is to improve (1.4) and obtain an asymptotic relation between the means defined by (1.1) and (1.2). In §2, the results are stated whereas their proofs are given in §3.

2. Statements of the main results and their discussions.

THEOREM 1. *If*

$$\frac{\overline{\lim}}{\underline{\lim}}_{r\to\infty}\frac{\log\log I_{\alpha,\beta}(r)}{\log r}=\frac{\mu}{\nu},\qquad (0\leq\nu\leq\mu\leq\infty),$$

then $\qquad \lambda\equiv\max(\lambda_f,\lambda_g)\leq\nu\leq\mu\leq\max(\rho_f,\rho_g)\equiv\rho.$

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THEOREM 2. *For any entire functions $f(z)$ and $g(z)$,*

$$\overline{\lim_{r\to\infty}}\left\{\frac{I_{\alpha,\beta}(r)}{I^{(k)}_{\alpha,\beta}(r)}\right\}^{\frac{1}{\log r}}=\frac{e^{\mu}}{e^{\nu}}.$$

REMARK. A part of this theorem includes (1.4) as a very special case.

THEOREM 3. *If* $\mu<\infty$, *then* $\log I_{\alpha,\beta}(r)\sim\log I^{(k)}_{\alpha,\beta}(r)$, *as* $r\to\infty$.

REMARK. If $f=g$ and $\alpha=\beta=\delta/2$, the result in theorem 3 reduces to a result for one function f, proved earliest by Kamthan [1] and Lakashminarashiman [3] (originally in a less general form due to Rahman [6]).

3. **Proof of theorem 1.** Following the arguments in ([4], p.420), for $R>r$, we have

$$I_{\alpha,\beta}(r)\le\left\{(M(r,f))^{\alpha}(M(r,g))^{\beta}\right\}^{\frac{1}{\alpha+\beta}}\le\left(\frac{R+r}{R-r}\right)^{\frac{1}{\alpha+\beta}}I_{\alpha,\beta}(R), \tag{3.1}$$

where $M(r,f)$ and $M(r,g)$ are the maximum modulii of $f(z)$ and $g(z)$, respectively, on $|z|=r$. The left hand side of this gives

$$\begin{aligned}\log I_{\alpha,\beta}(r)&<\frac{1}{\alpha+\beta}\left\{\alpha r^{\rho_f+\varepsilon}+\beta r^{\rho_g+\varepsilon}\right\}\\&=o(1)r^{\rho+\varepsilon}(1+o(1)r^{-\gamma}),\quad \gamma=|\rho_f-\rho_g|,\end{aligned}$$

for $r\ge r_0=r_0(\varepsilon)$, $\varepsilon>0$. Hence $\mu\le\rho$.

Further, taking $R=2r$ in the left hand inequality of (3.1), and proceeding similarly as above, it is established that $\nu\ge\lambda$.

This proves the theorem.

The proofs of theorems 2 & 3 needs the following lemma due to Kamthan([2], lemmas 5 & 6):

LEMMA. *Let $P(r)$ be a convex function with respect to $\varphi(r)$ in $(0,\infty)$, $\varphi(r)$ being an absolutely continuous and increasing function for $0<r<\infty$. Also, let*

$$\overline{\lim_{r\to\infty}}\frac{\log P(r)}{\varphi(r)}=\frac{A}{B},\quad(0\le B\le A\le\infty).$$

Then

$$\overline{\lim_{r\to\infty}}\exp\{(P(r)-N(r))(\varphi(r))^{-1}\}=\frac{e^{A}}{e^{B}}, \tag{3.2}$$

where $N(r)$ is a real valued function defined on $(0, \infty)$ *as*:

$$\exp\{N(r)+(k+1)\varphi(r)\}=\int_0^r \exp(P(x)+k\varphi(x))d\varphi(x).$$

Further, if $A<\infty$, *then*

(3.3) $$P(r)\sim N(r), \text{ as } r\to\infty.$$

Proofs of theorems 2 & 3. The function $|f(z)|^{\alpha}|g(z)|^{\beta}$ is of class PL ([5], p. 9) and so $\log I_{\alpha,\beta}(r)$ is a convex function of $\log r$ [5]. Therefore, in notations of the lemma above, if we take

$$P(r)=I_{\alpha,\beta}(r), \quad \varphi(r)=\log r,$$

then $A=\mu$, $B=\nu$ and $N(r)=\log I_{\alpha,\beta}^{(k)}(r)$.

Hence the results (3.2) and (3.3) in the lemma yield, respectively, the theorems 2 and 3.

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REFERENCES

[1] P.K. Kamthan. *Mean values of entire functions (II)*, Rev. Mate. His-Americana, Se. 4^a, no. 6, 24 (1964), 218-222.

[2] P.K. Kamthan. *On the comperative growth of convex functions*, Czechoslovak Mathematical Journal, 20 (95), (1970), 246-250.

[3] T.V. Lakshminarasimhan. *A note on the means of entire functions*, Proc. Amer. Math. Soc., 16, (1965), 1277-1279.

[4] ________, *On the mean of an entire function and the mean of the product of two entire functions*, Tohoku Mathematical Journal, 2nd series, no. 4, 19, (1967), 417-422.

[5] T. Rado. *Subharmonic functions*, Chalsea Pub., N.Y., 1949.

[6] Q. I. Rahman. *On the means of entire functions*, Quart. J. Math., 7, (1956), 192-195.

[7] ________, *On the means of entire functions (II)*, Proc. Amer. Math. Soc., 9, (1958), 748-750.