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# CONVEXITY THEOREM FOR $(N, p, q)$ SUMMABILITY

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**1.** Let $\sum_{n=0}^{\infty} a_n$ be an infinite series, and $\{s_n\}$ be the sequence of its partial sums i.e.,

$$s_n=\sum_{i=0}^{n} a_i.$$

For $\alpha$ real, define

$$A_0^{\alpha}=1, \quad A_n^{\alpha}=\frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{\underline{|n}} \qquad (n=1, 2, \cdots).$$

Let $\{p_n\}$ be a sequence with $p_0>0$ and $p_n\geq 0$ for $n>0$. Define

$$p_n^{\alpha}=\sum_{r=0}^{n} A_{n-r}^{\alpha-1} p_r. \tag{1.1}$$

The following identities are immediate:

$$\sum_{r=0}^{n} A_{n-r}^{\beta-1} p_r^{\alpha}=p_n^{\alpha+\beta}, \tag{1.2}$$

$$P_n^{\alpha}=p_n^{\alpha+1}=\sum_{r=0}^{n} p_r^{\alpha}, \tag{1.3}$$

where

$$P_n=\sum_{r=0}^{n} p_r.$$

Let $\{q_n\}$ be any sequence of constants, and write

$$(p*q)_n=p_0q_n+p_1q_{n-1}+\cdots+p_nq_0.$$

DEFINITION. $(N, p^{\alpha}, q)$ *summability.*

For $\alpha>-1$ and $\sum_{r=0}^{\infty} a_r$ a series, let

$$t_n^{\alpha}=\frac{1}{(p^{\alpha}*q)_n}\sum_{r=0}^{n} p_{n-r}^{\alpha} q_r s_r. \tag{1.4}$$

If $t_n^{\alpha}\to s$ as $n\to\infty$ we write

$$\sum_{r=0}^{\infty} a_r = s(N, p^\alpha, q) \text{ or } s_n \to s(N, p^\alpha, q).$$

If $t_n^\alpha = o(1)$ we write

$$\sum_{n=0}^{\infty} a_r \text{ is bounded } (N, p^\alpha, q).$$

REMARK. If we take $p_0=1$, $p_n=0$ for $n>0$ and $q_n=1$ for $n\geq 0$, then the above definition yields the standard definition of Cesaro summability of order $\alpha$.

2. The following theorem concerning Cesaro summability is known.

THEOREM A. *If* $\sum_{n=0}^{\infty} a_n$ *is bounded* $(c, \alpha)$ *and summable* $(c, \beta)$ *where* $\beta > \alpha > -1$, *then it is summable* $(c, r)$ *for every* $r > \alpha$.

A proof of this theorem is given in Zygmund [4], and for the case where $\alpha$, $\beta$ and $r$ are integers in Hardy ([2], Theorem 70).

The purpose of this paper is to extend the scope of this theorem to include certain other families of $(N, p, q)$ methods of summability. The theorem which we establish here includes theorem A as a particular case.

3. In order to prove our theorem we requires some restriction on the sequence $(p^\xi * q)_n$. We shall impose the following condition:

For each $\xi > -1$ there exist positive constants $H_1$ and $H_2$ (which may depend on $\xi$ but not on $n$) such that

$$H_1 n^\xi \leq (p^\xi * q)_n / (p * q)_n \leq H_2 n^\xi. \tag{3.1}$$

It may be remarked that (3.1) does not hold in general.

THEOREM. *If* $\beta > r > \alpha > -1$, (3.1) *holds for* $\xi > -1$, $\sum_{n=0}^{\infty} a_n$ *is bounded* $(N, p^\alpha, q)$, *and summable* $(N, p^\beta, q)$, *then* $\sum_{n=0}^{\infty} a_n$ *is summable* $(N, p^r, q)$.

In view of ([3], Theorem 1), the above theorem is a consequence of the following lemma.

LEMMA. *If* $\alpha > -1$, (3.1) *holds for* $\xi > -1$, $\sum_{r=0}^{\infty} a_r$ *is bounded* $(N, p^\alpha, q)$, *and summable* $(N, p^{\alpha+1}, q)$ *to zero, and* $0 < \delta < 1$, *then* $\sum_{r=0}^{\infty} a_r$ *is summable* $(N, p^{\alpha+\delta}, q)$

*to zero.*

PROOF. Let $T_n^{(r)}=\sum_{i=0}^{n} p_{n-i}^{r} q_i s_i$. We are given that

$$T_n^{(\alpha)}=o((p^{\alpha}*q)_n),$$

and

$$T_n^{(\alpha+1)}=o((p^{\alpha+1}*q)_n).$$

We are required to prove that

$$T_n^{(\alpha+\delta)}=o((p^{\alpha+\delta}*q)_n).$$

Now

$$T_n^{(\alpha+\delta)}=\sum_{i=0}^{n} A_{n-i}^{\delta-1} T_i^{(\alpha)}.$$

Thus for $Q$ in the open interval $\left(\frac{1}{2}, 1\right)$ we have

$$T_n^{(\alpha+\delta)}=\sum_{i=0}^{[Qn]} A_{n-i}^{\delta-1} T_i^{(\alpha)}+\sum_{i=[Qn]+1}^{n} A_{n-i}^{\delta-1} T_i^{(\alpha)}$$
$$=I_1+I_2 \text{ say.}$$

Let us consider $I_1$ first. By Abel's formula for partial summation we have

$$I_1=\sum_{i=0}^{[Qn]-1} A_{n-i}^{\delta-2} T_i^{(\alpha+1)}+A_{n-[Qn]}^{\delta-1} T_{[Qn]}^{(\alpha+1)}.$$

So

$$|I_1|\leq\sum_{i=0}^{[Qn]} |A_{n-i}^{\delta-2}|\,|T_i^{(\alpha+1)}|+o(\{n-[Qn]\}^{\delta-1}(p^{\alpha+1}*q)n).$$

Now by the condition (3.1) we find that

$$\frac{|I_1|}{(p^{\alpha+\delta}*q)_n}\leq\frac{(p^{\alpha+1}*q)_n}{(p^{\alpha+\delta}*q)_n}\sum_{i=0}^{[Qn]}|A_{n-i}^{\delta-2}|\eta_i+o(1)\,;$$

where $\eta_i\geq 0$ and tends to zero as $i$ tends to infinity. Again using (3.1) we obtain

$$|I_1|/(p^{\alpha+\delta}*q)_n=o(n^{1-\delta}[\sum_{i=0}^{[Qn]}|A_{n-i}^{\delta-2}|\eta_i])+o(1).$$

Now it is easy to see that

$$|I_1|=o((p^{\alpha+\delta}*q)_n).$$

Consider

$$|I_2|\leq\sum_{i=[Qn]+1}^{n} A_{n-i}^{\delta-1}|T_i^{(\alpha)}|=o\Big(\max_{[Qn]+1\leq i\leq n}(p^{\alpha}*q)_i \sum_{i=[Qn]+1}^{n} A_{n-i}^{\delta-1}\Big)$$

$$=o((p^{\alpha}*q)_n \sum_{i=1}^{n-[Qn]} i^{\delta-1})$$

$$=o((p^{\alpha}*q)_n\{n-[Qn]\}^{\delta}).$$

Again using (3.1) we find that

$$|I_2|/(p^{\alpha+\delta}*q)_n=o(\{1-Q\}^{\delta}).$$

Thus

$$\limsup_{n\to\infty}(|T_n^{(\alpha+\delta)}|/(p^{\alpha+\delta}*q)_n)\leq\limsup_{n\to\infty}(|I_1|/(p^{\alpha+\delta}*q)_n+|I_2|/(p^{\alpha+\delta}*q)_n).$$

$$\leq H(1-Q)^{\delta}.$$

Since Q is any number in the interval $\left(\frac{1}{2}, 1\right)$ it follows that

$$T_n^{\alpha+\delta}=o((p^{\alpha+\delta}*q)_n) \text{ as required.}$$

It may be remarked that for $q_n=1$, $n=0, 1, \cdots$, our theorem reduces to the theorem of Cass [1].

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