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A NOTE ON AN INTERSECTION THEOREM IN BANACH SPACE

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Introduction.

Let X be a real Banach Space. Assume that $X=A\oplus B$, i.e. X is the direct sum of two subspaces $A\subset X$ and $B\subset X$.

Let $f: A\rightarrow X$ and $g: B\rightarrow X$ and $h: A\rightarrow X$ be three continuous mappings. We have given conditions on f, g, and h such that $f(A)\cap g(B)\cap h(A)\neq\phi$.

1. In [1] Kannan has given a theorem concerning simultaneous fixed points of two mappings in a metric space. We give here a version of the theorem in a Banach Space.

THEOREM A. *Let E be a Banach Space. If T_1 and T_2 are continuous mappings from E into E such that*

$$\|T_1(x)-T_2(y)\|\leq\beta\{\|x-T_1(x)\|+\|y-T_2(y)\|\}$$

where $x,y\in E$ and $0<\beta<\frac{1}{2}$, then T_1 and T_2 have a common fixed point.

2. An intersection theorem in Banach Space.

By $Q_A: X\rightarrow A$ and $Q_B: X\rightarrow B$ we denote the projection mappings of X onto A and X onto B respectively. The mappings Q_A and Q_B are linear and we have

$$\|Q_Ax\|\leq\|Q_A\|\|x\|,\quad \|Q_Bx\|<\|Q_B\|\|x\|$$

for all x in X, where $\|Q_A\|$ and $\|Q_B\|$ are the norms of Q_A and Q_B respectively. Let F, G and H are mappings from X into X and define the mappings T_1 and T_2 as follows:

$$T_1=F\circ Q_A-G\circ Q_B$$
$$T_2=H\circ Q_A-G\circ Q_B.$$

THEOREM 1. *Let $X=A\oplus B$ and let $f: A\rightarrow X$, $g: B\rightarrow X$* and *$h: A\rightarrow X$* be such that

$$f(a)=a-F(a), \text{ for all } a \text{ in } A.$$

$$g(b)=b-G(b), \text{ for all } b \text{ in } B$$
$$h(a)=a-H(a), \text{ for all } a \text{ in } A$$

and also T_1 and T_2 satisfy the hypothesis of theorem A above. Then $f(A)\cap g(B)\cap h(A)\neq\phi$.

Proof of theorem 1.

Since $X=A\oplus B$, any element x in X can be written as $x=a-b$ where $a\in A$ and $b\in B$. Since from theorem A, T_1 and T_2 have a common fixed point which is x_0 say with representation $x_0=a_0-b_0$, we have,

$$T_1(x_0)=x_0$$

i.e. $$F\circ Q_A(x_0)-G\circ Q_B(x_0)=a_0-b_0$$

i.e. $$F(a_0)-G(b_0)=a_0-b_0$$

in other words

$$a_0-F(a_0)=b_0-G(b_0)$$

also we have

$$T_2(x_0)=x_0$$

i.e. $$H\circ Q_A(x_0)-G\circ Q_B(x_0)=a_0-b_0$$

i.e. $$H(a_0)-G(b_0)=a_0-b_0$$

i.e. $$a_0-H(a_0)=b_0-G(b_0)$$

and therefore $$f(A)\cap g(B)\cap h(A)\neq\phi.$$

In case f, g and h are linear mappings the conditions on f, g and h can be given in terms of the norms of the linear mappings F, G and H to make sure that $f(A)\cap g(B)\cap h(A)\neq\phi$.

THEOREM 2. *Let $X=A\oplus B$ and let f, g and h are linear continuous mappings as shown in respective subspaces of theorem* 1. *Then if*

$$\|F\|\|Q_A\|+\|G\|\|Q_B\|<\frac{\beta}{1+\beta}$$

and (1)

$$\|H\|\|Q_A\|+\|G\|\|Q_B\|<\frac{\beta}{1+\beta}$$

then $$f(A)\cap g(B)\cap h(A)\neq\phi.$$

Proof of theorem 2.

Proof lies in observing the fact that the conditions in (1) imply that $F\circ Q_A-G\circ Q_B$ and $H\circ Q_A-G\circ Q_B$ satisfy the hypothesis of theorem A. We start with the inequality

$$\|F\circ Q_A(x)-G\circ Q_B(x)-H\circ Q_A(y)+G\circ Q_B(y)\|$$
$$\leq\beta\{\|x-F\circ Q_A(x)+G\circ Q_B(x)\|+\|y-H\circ Q_A(y)+G\circ Q_B(y)\|\} \quad (2)$$

Now left hand side of (2) is

$$\leq(\|F\|\|Q_A\|+\|G\|\|Q_B\|)\|x\|+(\|H\|\|Q_A\|+\|G\|\|Q_B\|)\|y\|$$

also

$$\beta\{\|x-F\circ Q_A(x)\|-\|G\circ Q_B(x)\|\}+\beta\{\|y-H\circ Q_A(y)\|-\|G\circ Q_B(y)\|\leq\text{R. H. S.}$$

since

$$\|G\circ Q_B(x)\|\leq\|G\|\|Q_B\|\|x\|, \text{ for all } x \text{ in } X,$$

we have

$$\beta\{\|x-F\circ Q_A(x)\|-\|G\|\|Q_B\|\|x\|\}+\beta\{\|y-H\circ Q_A(y)\|-\|G\|\|Q_B\|\|y\|\}\leq\text{R. H. S.}$$

from this we get

$$\beta\{\|x\|-\|F\circ Q_A(x)\|-\|G\|\|Q_B\|\|x\|\}+\beta\{\|y\|-\|H\circ Q_A(y)\|-\|G\|\|Q_B\|\|y\|\}\leq\text{R. H. S.}$$

therefore

$$\beta[\|x\|\{(1-\|G\|\|Q_B\|)-\|F\|\|Q_A\|\}]+\beta[\|y\|\{(1-\|G\|\|Q_B\|)-\|H\|\|Q_A\|\}]\leq\text{R. H. S.}$$

therefore if

$$\|F\|\|Q_A\|+\|G\|\|Q_B\|\leq\beta[(1-\|G\|\|Q_B\|)-\|F\|\|Q_A\|]$$

and

$$\|H\|\|Q_A\|+\|G\|\|Q_B\|\leq\beta[(1-\|G\|\|Q_B\|)-\|H\|\|Q_A\|] \quad (3)$$

then (2) is satisfied.

The conditions (3) can be written as

$$\|F\|\|Q_A\|+\|G\|\|Q_B\|\leq\frac{\beta}{1+\beta}$$

and
$$\|H\|\|Q_A\|+\|G\|\|Q_B\|\leq\frac{\beta}{1+\beta}$$

which completes the proof.

COROLLARY to theorem 2.

Let R denote a set of real numbers. Let $X=A\oplus B$ and $f: R\times A\to X$, $g: R\times B\to X$ and $h: R\times A\to X$ be continuous linear mappings such that

$$f(\lambda : a)=a-\lambda F(a) \text{ for all } a \text{ in } A$$

and λ is in R,

$$g(\lambda : b)=b-\lambda G(b) \text{ for all } b \text{ in } B$$
$$h(\lambda : a)=a-\lambda H(a) \text{ for all } a \text{ in } A.$$

Then if

$$|\lambda|\{\|F\|\|Q_A\|+\|G\|\|Q_B\|\}\leq\frac{\beta}{1+\beta}$$

and $$|\lambda| \{\|H\|\|Q_A\|+\|G\|\|Q_B\|\} \leq \frac{\beta}{1+\beta}$$

then $$f(\lambda : A) \cap g(\lambda : B) \cap h(\lambda : A) \neq \phi.$$

The proof is similar to the proof of theorem 2.

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