

MILDLY NORMAL SPACES

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The concept of almost normal spaces has been introduced recently in [5]. A space is said to be *almost normal* if every pair of disjoint sets, one of which is closed and the other is regularly closed, can be strongly separated. Obviously, every normal space is almost normal. Also, there exist almost normal spaces which are not normal. In the present paper, we propose to introduce the class of mildly normal spaces which properly contains the class of almost normal spaces. Mildly normal spaces have also been studied recently by E.P. Lane [2] in connection with insertion of continuous functions.

DEFINITION 1 A space X is said to be *mildly normal* if for every pair of disjoint regularly closed subsets F_1 and F_2 of X , there exist disjoint open sets U and V such that $F_1 \subset U$, $F_2 \subset V$.

Obviously, every almost normal space is mildly normal. However, a mildly normal space may fail to be almost normal as is shown by the following example.

EXAMPLE 1 Let $X = \{a_{ij}, c_i, a : i, j=1, 2, \dots\}$. Each a_{ij} is isolated. A fundamental system of neighbourhoods of c_i is $\{U^n(c_i) : n=1, 2, \dots\}$ where $U^n(c_i) = \{c_i, a_{ij} : j \geq n\}$ and that of a is $\{U^n(a) : n=1, 2, \dots\}$ where $U^n(a) = \{a, a_{ij} : i, j \geq n\}$. It may be verified that this space is a Hausdorff space which is mildly normal but not almost normal.

THEOREM 1 For a space X , the following are equivalent:

- (a) X is mildly normal.
- (b) For every regularly closed set A and every regularly open set B containing A , there is an open set V such that $A \subset V \subset \bar{V} \subset B$.
- (c) For every regularly open set B containing a regularly closed set A , there exists a regularly open set U such that $A \subset U \subset \bar{U} \subset B$.
- (d) For every pair of disjoint regularly closed sets A and B , there exist open sets U and V such that $A \subset U$, $B \subset V$ and $\bar{U} \cap \bar{V} = \phi$.

PROOF. (a) \Rightarrow (b). If B is a regularly open set containing a regularly closed set A , then A and $X \sim B$ are disjoint regularly closed subsets of X . Therefore, there exist disjoint open sets U and V such that $A \subset V$, $X \sim B \subset U$. It follows that $X \sim U \subset B$ and thus $A \subset V \subset \bar{V} \subset B$.

(b) \Rightarrow (c). If B be a regularly open set containing a regularly closed set A , then there exists an open set V such that $A \subset V \subset \bar{V} \subset B$. If $\bar{V}^\circ = U$, then $A \subset U \subset \bar{U} \subset B$ where U is regularly open.

(c) \Rightarrow (d). If A and B be disjoint regularly closed sets, then $X \sim B$ is a regularly open set containing the regularly closed set A . Therefore there exists a regularly open set W such that $A \subset W \subset \bar{W} \subset X \sim B$. Again, since W is a regularly open set containing the regularly closed set A , therefore there exists a regularly open set U such that $A \subset U \subset \bar{U} \subset W$. Let $X \sim \bar{W} = V$. Then, $A \subset U$, $B \subset V$ and $\bar{U} \cap \bar{V} = \phi$.

(d) \Rightarrow (a). Obvious.

THEOREM 2. *Every closed, continuous, open image of a mildly normal space is mildly normal.*

PROOF. Easy to verify.

THEOREM 3. *A space X is mildly normal if and only if for every pair of disjoint regularly closed sets A and B , there exists a continuous function f on X into the closed interval $[0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

PROOF. First, if A and B be disjoint regularly closed sets and if f be the continuous function on X into $[0, 1]$ such that $f(A) = \{0\}$, $f(B) = \{1\}$, then the sets $f^{-1}\{[0, 1/2[\}$ and $f^{-1}\{]1/2, 1\}$ are disjoint open sets containing A and B respectively. Hence X is mildly normal.

Conversely, let X be mildly normal and let A and B be disjoint regularly closed subsets of X . Let $X \sim B = G_1$. Then G_1 is a regularly open set containing the regularly closed set A . Therefore, in view of theorem 1. (c), there exists a regularly open set $G_{1/2}$ such that $A \subset G_{1/2} \subset \bar{G}_{1/2} \subset G_1$. If the sets $G_{p/2^i}$, $p=1, \dots, 2^i$ have been determined such that $A \subset G_{1/2^i} \subset \bar{G}_{1/2^i} \subset G_{2/2^i} \subset \dots \subset \bar{G}_{(p-1)/2^i} \subset G_{p/2^i} \subset \dots \subset G_1$, then since X is mildly normal, there exists $G_{(2j+1)/2^{i+1}}$ such that $A \subset G_{1/2^{i+1}} \subset \bar{G}_{1/2^{i+1}} \subset G_{2/2^{i+1}}$ and $\bar{G}_{2j/2^{i+1}} \subset G_{(2j+1)/2^{i+1}} \subset \bar{G}_{(2j+1)/2^{i+1}} \subset G_{(2j+2)/2^{i+1}}$, $1 \leq j \leq 2^i - 1$. By induction it follows that if Q be the set of all rationals of the form $j/2^i$, $1 \leq j \leq 2^i$, i being any natural number, then the sets G_r may be defined for all $r \in Q$ such that if r_1 and r_2 be any two rationals in Q where $r_1 < r_2$, then $A \subset$

$G_{r_1} \subset \bar{G}_{r_1} \subset G_{r_2} \subset \bar{G}_{r_2} \subset G_1$. Let f be a function defined on X into $[0,1]$ such that $f(x)=1$ for all $x \in B$ and $f(x)=\text{glb}\{r : r \in Q, x \in X \sim G_r\}$ for all $x \in G_1$. Then $f(A)=\{0\}$, $f(B)=\{1\}$. That f is continuous may be verified as in the proof of the standard Urysohn's Lemma.

THEOREM 4 *A space X is mildly normal if and only if for every function f on a regularly closed set A into $[0,1]$, there exists a continuous function $f^* : X \rightarrow [0,1]$ such that $f^*/A=f$.*

PROOF. Using theorem 3 above and observing that every regularly closed subset of a regularly closed subset of X is itself a regularly closed subset of X [1, page 75], the proof involves standard techniques used in the proof of the well known Tietze extension theorem for normal spaces.

DEFINITION 2 [Singal and Arya, 4]. A space X is said to be *almost regular* if for every regularly closed set A and a point $x \notin A$, there exists open sets U and V such that $A \subset U$, $x \in V$ and $U \cap V = \phi$.

THEOREM 5 *Every almost regular, almost compact space is mildly normal.*

PROOF. Let A and B be disjoint regularly closed subsets of an almost regular, almost compact space X . X being almost regular, for each $x \in A$, there exist open sets G_x and H_{Bx} such that $x \in G_x$, $B \subset H_{Bx}$ and $\bar{G}_x \cap \bar{H}_{Bx} = \phi$. Then, $\{G_x \cap A : x \in A\}$ is a relatively open covering of A . Since every regularly closed subset of an almost compact space is almost compact, therefore A is almost compact. It follows that there exists a finite subfamily $\{G_x \cap A : i=1, \dots, n\}$ whose closures (in A) cover A . Obviously then $A \subset \bigcup_{i=1}^n \bar{G}_{x_i}$. Let $H = \bigcap_{i=1}^n H_{Bx_i}$, and let $G = X \sim \bigcap_{i=1}^n \bar{H}_{Bx_i}$. Also, $A \subset G$, $B \subset H$ and $G \cap H = \phi$. Hence X is mildly normal.

COROLLARY 1 [Papić, 3]. *Every almost compact Urysohn space is mildly normal.*

PROOF. Every almost compact Urysohn space is almost regular [3].

THEOREM 6 *Every almost regular, Lindelöf space is mildly normal.*

PROOF. Let X be an almost regular, Lindelöf space and let A and B be two disjoint regularly closed subsets of X . For each $x \in A$, there exists an open set U_x such that $x \in U_x \subset \bar{U}_x \subset X \sim B$. It follows that for each point $x \in A$, there is an open set U_x such that $x \in U_x$ and $\bar{U}_x \cap B = \phi$. Then $\mathcal{U} = \{U_x : x \in A\}$ is an open covering of A . Since every closed subset of a Lindelöf space is Lindelöf, therefore \mathcal{U} admits of a countable subcovering $\{U_n : n=1, 2, \dots\}$. Similarly, for each point

$y \in B$, there exists an open set V_y such that $y \in V_y$ and $\bar{V}_y \cap A = \phi$. Again $\mathcal{V} = \{V_y : y \in B\}$ is an open covering of the Lindelöf set B and therefore \mathcal{V} has a countable subcovering $\{V_n : n=1, 2, \dots\}$. Let $A_n = U_n \sim \cup \{\bar{V}_k : k \leq n\}$ and let $B_n = V_n \sim \cup \{\bar{U}_k : k \leq n\}$ for each $n=1, 2, \dots$. Since $A_n \cap V_m = \phi$ for all $n \geq m$, therefore $A_n \cap B_m = \phi$ for all $n \geq m$. Similarly, $A_n \cap B_m = \phi$ for all $n \leq m$. Hence $A_n \cap B_m = \phi$ for all m, n . If $G = \cup \{A_n : n=1, 2, \dots\}$ and $H = \cup \{B_n : n=1, 2, \dots\}$, then G and H are disjoint open sets such that $A \subset G, B \subset H$. Hence X is mildly normal.

THEOREM 7 *Every almost regular space with a σ -locally finite base is mildly normal.*

PROOF. Let F_1 and F_2 be disjoint regularly closed subsets of an almost regular space with a σ -locally finite base $\{G_{n(\lambda)} : n=1, 2, \dots, \lambda \in \Lambda_n\}$. Since X is almost regular, therefore for each point $x \in F_1$, there exists a basic open set $G_{n_x(\lambda_x)}$ containing x whose closure is contained in $X \sim F_2$ and for each $y \in F_2$, there is a basic open set $G_{n_y(\lambda_y)}$ containing y whose closure is contained in $X \sim F_1$. Let $G_{k(F_1)} = \cup_{x \in F_1} G_{k(\lambda_x)}$ and $G_{k(F_2)} = \cup_{y \in F_2} G_{k(\lambda_y)}$. Then $\bar{G}_{k(F_1)} = \cup_{x \in F_1} \bar{G}_{k(\lambda_x)} \subset X \sim F_2$, and $\bar{G}_{k(F_2)} = \cup_{y \in F_2} \bar{G}_{k(\lambda_y)} \subset X \sim F_1$. Let $U_{n(F_1)} = G_{n(F_1)} \sim \cup_{k \leq n} \bar{G}_{k(F_2)}$ and let $U_{n(F_2)} = G_{n(F_2)} \sim \cup_{k \leq n} \bar{G}_{k(F_1)}$. If $U_{F_1} = \bigcup_{n=1}^{\infty} U_{n(F_1)}$ and $U_{F_2} = \bigcup_{n=1}^{\infty} U_{n(F_2)}$ then $F_1 \subset U_{F_1}$, $F_2 \subset U_{F_2}$ and $U_{F_1} \cap U_{F_2} = \phi$. Also, U_{F_1} and U_{F_2} are open. Hence X is mildly normal.

DEFINITION 3 [Singal and Arya, 4]. A space X is said to be *weakly regular* if for every point x and every regularly open set U containing x , there is an open set V such that $x \in V \subset \bar{V} \subset U$.

THEOREM 8 *Every weakly regular nearly paracompact space is mildly normal.*

PROOF. Let X be a weakly regular, nearly paracompact space and let A and B be any two disjoint regularly closed subsets of X . Let $x \in A$. Then $\{x\} \subset A \subset X \sim B$. Since X is weakly regular, there exists an open set V_x such that $\bar{V}_x \cap B = \phi$ and $x \in V_x$. Then $\{\bar{V}_x^o : x \in A\} \cup \{X \sim A\}$ is a regular open covering of X . Since X is nearly paracompact, therefore this covering has a locally finite open refinement. Let $\mathcal{Z} = \{U_\alpha : \alpha \in \Lambda\}$ be the family of those members of this refinement which intersect A . Let $U = \cup \{U_\alpha : \alpha \in \Lambda\}$. Then U is an open set containing A . Let $U^* = X \sim \cup \{\bar{U}_\alpha : \alpha \in \Lambda\}$. Then U^* is an open set such that $U \cap U^* = \phi$. For each $\alpha \in \Lambda$, there exists an $x \in A$ such that $U_\alpha \subset \bar{V}_x^o$. Since $\bar{U}_\alpha \subset \bar{V}_x^o = \bar{V}_x \subset X \sim B$, therefore B

$\bigcap \bar{U}_\alpha = \phi$ for all α . Thus $B \subset X \sim \bigcup \{\bar{U}_\alpha : \alpha \in \Lambda\} = W$ (say).

Then U and W are disjoint open sets containing A and B respectively.

COROLLARY 2 *Every almost regular, nearly paracompact space is mildly normal.*

PROOF. Every almost regular space is weakly regular.

COROLLARY 3 *Every nearly paracompact Hausdorff space is mildly normal.*

PROOF. Every nearly paracompact Hausdorff space is almost regular.

THEOREM 9 *Every regularly closed subspace of a mildly normal space is mildly normal.*

PROOF. Follows easily in view of the fact that if F be a regularly closed subset of a regularly closed subset Y of X , then F is a regularly closed subset of X .

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