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INTEGRALS INVOLVING PRODUCTS OF GENERALIZED LEGENDRE FUNCTIONS AND THE H-FUNCTION

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1. Introduction. In this paper, we evaluate some integrals involving products of generalized Legendre functions and Fox's H-function. On specializing the parameters of these functions in the integrals, we can get many new as well as known results as particular cases.

Generalized Legendre functions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$, solutions of the differential equation:

$$(1.1)\quad (1-z^2)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}\left\{k(k+1)-\frac{m^2}{2(1-z)}-\frac{n^2}{2(1+z)}\right\}w=0,$$

introduced in [4] by Kuipers and Meulenbeld, have been defined for all points of the z-plane in which a cross-cut exists along the real axis from 1 to $-\infty$ and in [5] for the real values of z on the cross-cut for $-1<z<1$. These functions reduce to associated Legendre functions on setting $m=n$ and to Legendre functions on putting $m=n=0$.

The H-function has been introduced by Fox [3, p. 408], and its conditions of validity, asymptotic expansions and analytic continuations have been discussed by Braaksma [1]. Following the definition given by Braaksma [1, pp. 239—241], it will be represented as follows:

$$(1.2)\quad H_{r,s}^{l,u}\left[z\,\middle|\,\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right]=\frac{1}{2\pi i}\int_T\frac{\prod_{j=1}^{l}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{u}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=l+1}^{s}\Gamma(1-b_j+\beta_j\xi)\prod_{j=u+1}^{r}\Gamma(a_j-\alpha_j\xi)}Z^{\xi}\,d\xi$$

where $\{(a_r,\alpha_r)\}$, stands for the set of parameters $(a_1,\alpha_1),\cdots,(a_r,\alpha_r)$.

In what follows, for the sake of brevity

$$\sum_1^s(\beta_j)-\sum_1^r(\alpha_j)\equiv A \text{ and } \sum_1^l(\beta_j)-\sum_{l+1}^s(\beta_j)+\sum_1^u(\alpha_j)-\sum_{u+1}^r(\alpha_j)\equiv B.$$

2. The integrals to be established are as under:

$$(2.1)\quad \int_0^1 x^p(1-x)^q(1-zx)^{-\frac{n}{2}} P^{m,n}_{k-\frac{m-n}{2}}(1-2xz)\, H^{l,u}_{r,s}\left[yx^\gamma(1-x)^\delta \left| \begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right.\right] dx$$

$$=2^{\frac{n-m}{2}} z^{-\frac{1}{2}m} \sum_{N=0}^{k} \frac{z^N(1+k-m+n)_N(-k)_N}{N!\Gamma(1-m+N)}\times$$

$$\times H^{l,u+2}_{r+2,s+1}\left[y\left|\begin{matrix}(-q,\ \delta),\ \left(-p-N+\frac{m}{2},\ \gamma\right),\ \{(a_r,\ \alpha_r)\}\\ \{(b_s,\beta_s)\},\ \left(-1-p-q+\frac{m}{2},\ \gamma+\delta\right)\end{matrix}\right.\right],$$

where $A\geq 0$, $B>0$, $|\arg y|<\frac{1}{2}B\pi$, z real, $|z|<1$, $\mathrm{Re}(p+\gamma\cdot b_j/\beta_j)>\frac{1}{2}\ \mathrm{Re}(m)-1$, $\mathrm{Re}(q+\delta\cdot b_j/\beta_j)>-1$ $(j=1,\ 2,\cdots,l)$ and $\gamma>0$, $\delta\geq 0$ (or $\gamma\geq 0$, $\delta>0$).

$$(2.2)\quad \int_{-1}^1 (1-x)^{-\frac{m}{2}}(1+x)^q(z+x)^{-\sigma} P^{m,n}_k(x)\, H^{l,u}_{r,s}\left[y(1+x)^\delta(z+x)^{-\mu}\left|\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right.\right]dx$$

$$=2^{1+q-m+\frac{1}{2}n}(z-1)^{-\sigma}\sum_{N=0}^{\infty}\frac{2^N}{N!(1-z)^N}\times$$

$$\times H^{l,u+3}_{r+3,s+3}\left[\frac{2^\delta y}{(z-1)^\mu}\left|\begin{matrix}\left(-q-\frac{n}{2}-N,\delta\right),\ (1-\sigma-N,\mu),\ \left(-q-N+\frac{n}{2},\delta\right),\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\},(1-\sigma,\mu),\left(k-q-N+\frac{m}{2},\delta\right),\left(-1-q-k-N+\frac{m}{2},\delta\right)\end{matrix}\right.\right],$$

where $A\geq 0, B>0$, $|\arg y|<\frac{1}{2}B\pi$, z not lying on the cut $(-1,+1)$, $\mathrm{Re}(m)<1$, $\mathrm{Re}(q+1+\delta\cdot b_j/\beta_j)>\frac{1}{2}|\mathrm{Re}\, n|$ $(j=1,\ 2,\cdots,l)$ and $\delta>0$, $\mu\geq 0$ (or $\delta\geq 0$, $\mu>0$).

$$(2.3)\quad \int_0^\infty e^{-pt}(1-e^{-t})^{\frac{1}{2}m}(1+e^{-t})^\sigma P^{-m,-n}_k(e^t)\, H^{l,u}_{r,s}\left[z(1+e^{-t})^\delta\left|\begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right.\right]dx$$

$$=2^{-p+\frac{1}{2}(m-n)}\sum_{N=0}^{\infty}\frac{\Gamma(p-k+N)\ \Gamma(p+k+N+1)}{2^N N!\Gamma\left(p+\frac{1}{2}(m-n)+1+N\right)\Gamma\left(p+\frac{1}{2}(m+n)+1+N\right)}\times$$

$$\times H^{l,u+1}_{r+1,s+1}\left[z\left|\begin{matrix}\left(-\sigma-p-N-\frac{m}{2},\ \delta\right),\ \{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\},\ \left(-\sigma-p-\frac{m}{2},\ \delta\right)\end{matrix}\right.\right],$$

where $\delta>0$, $A\geq 0$, $B>0$, $|\arg z|<\frac{1}{2}B\pi$, $\mathrm{Re}(m)>-1$, $\mathrm{Re}(p)>-\mathrm{Re}(k)-1$,

$\mathrm{Re}(p)>\mathrm{Re}(k)$.

$$(2.4)\quad \int_0^\infty e^{-pt}(e^t-1)^{\frac{1}{2}m}\left(\frac{\rho e^t}{\rho-2}-1\right)^{\sigma} P_k^{-m,-n}(\rho e^t-\rho+1)\times$$

$$\times H_{r,s}^{l,u}\left[z\left(\frac{\rho e^t}{\rho-2}-1\right)^{\delta}\,\middle|\,\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right]dx$$

$$=\rho^{p-\frac{m}{2}}(\rho-2)^{-\sigma}\sum_{N=0}^{\infty}\frac{(2-\rho)^N(p+1)_N}{2^N\,N!}\times$$

$$\times H_{r+2,s+2}^{l+2,u}\left[z(\rho-2)^{-\delta}\,\middle|\,\begin{matrix}\{(a_r,\alpha_r)\},\left(p-\sigma+N+1-\frac{n}{2},\ \delta\right),\ \left(p-\sigma+1+N+\frac{n}{2},\ \delta\right)\\ \left(p-\sigma-k+N-\frac{m}{2},\ \delta\right),\ \left(p-\sigma+k+1+N-\frac{m}{2},\ \delta\right),\ \{(b_s,\beta_s)\}\end{matrix}\right]$$

provided $\delta>0$, $A\geq 0$, $B>0$, $|\arg z|<\frac{1}{2}B\pi$, $\mathrm{Re}(\rho)>0$, $\mathrm{Re}(m)>-1$, $\mathrm{Re}(p)>\mathrm{Re}\left(\frac{1}{2}m+\sigma-k+\delta(a_i-1)/\alpha_i\right)$, $\mathrm{Re}(p)>\mathrm{Re}\left(\frac{m}{2}+\sigma+k+\delta(a_i-1)/\alpha_i\right)$ $(i=1,2,\cdots,u)$.

PROOF. To establish the integral (2.1), expressing the H-function in the integrand as Mellin-Barnes type of integral (1,2), interchanging the order of integration which is justifiable due to the absolute convergence of the integrals involved in the process, we get

$$(2.5)\quad \frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{l}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{u}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=l+1}^{s}\Gamma(1-b_j+\beta_j\xi)\prod_{j=u+1}^{r}\Gamma(a_j-\alpha_j\xi)}y^{\xi}\times$$

$$\times\int_0^1 x^{p+\gamma\xi}(1-x)^{q+\delta\xi}(1-zx)^{-\frac{n}{2}}P_{k-\frac{m-n}{2}}^{m,n}(1-2xz)\,dx\,d\xi.$$

Evaluating the inner integral with the help of [6, p. 287(9)], i.e.

$$(2.6)\quad \int_0^1 x^p(1-x)^q(1-zx)^{-\frac{1}{2}n}P_k^{m,n}(1-2xz)\,dx$$

$$=2^{\frac{1}{2}(n-m)}z^{-\frac{1}{2}m}\frac{\Gamma\left(p-\frac{m}{2}+1\right)\Gamma(q+1)}{\Gamma(1-m)\,\Gamma\left(p+q-\frac{1}{2}m+2\right)}\times$$

$$\times{}_3F_2\left(\beta+1,\ -\gamma,\ p-\frac{1}{2}m+1;\ 1-m,\ p+q-\frac{1}{2}m+2;\ z\right),$$

where $\beta=k-\frac{1}{2}(m-n)$, $\gamma=k+\frac{1}{2}(m-n)$, $\text{Re}(p)>\frac{1}{2}\text{Re}(m)-1$, $\text{Re}(q)>-1$, z real, $|z|<1$; then expressing the hypergeometric function as series and changing the order of summation and integration in view of [2, p. 176(75)], which is permissible under the conditions given in (2.1) and (2.6) ; and again applying (1.2), the definition of the H-function, the value of the integral is obtained.

Proceeding on similar lines, the results (2.2) to (2.4) can be established with the help of [6, pp. 286—287(13), (5) & (6)], viz.,

$$(2.7)\quad \int_{-1}^{1}(1-x)^{-\frac{1}{2}m}(1+x)^{q}(u+x)^{-\sigma}P_{k}^{m,n}(x)=$$

$$=2^{q+\frac{1}{2}n-m+1}\frac{\Gamma\left(q+\frac{1}{2}n+1\right)\Gamma\left(q-\frac{1}{2}n+1\right)}{\Gamma\left(q-k-\frac{1}{2}m+1\right)\Gamma\left(q+k-\frac{1}{2}m+2\right)}(u-1)^{-\sigma}\times$$

$$\times{}_3F_2\left(q+\frac{1}{2}n+1,\ \sigma,\ q-\frac{1}{2}n+1\ ;\ q-k-\frac{1}{2}m+1,\ q+k-\frac{1}{2}m+2\ ;\ \frac{2}{1-u}\right),$$

where $\text{Re}(m)<1$, $\text{Re}(q+1)>\frac{1}{2}|\text{Re}\ n|$, u not lying on the cut $(-1,+1)$:

$$(2.8)\quad \int_{0}^{\infty}e^{-pt}(1-e^{-t})^{\frac{1}{2}m}(1+e^{-t})^{\sigma}P_{k}^{-m,-n}(e^{t})dt$$

$$=\frac{2^{-p+\frac{1}{2}(m-n)}\Gamma(p-k)\Gamma(p+k+1)}{\Gamma\left(p+\frac{1}{2}m-\frac{1}{2}n+1\right)\Gamma\left(p+\frac{1}{2}m+\frac{1}{2}n+1\right)}\times$$

$$\times{}_3F_2\left(p-k,\ p+\frac{1}{2}m+\sigma+1,\ p+k+1\ ;\ p+\frac{1}{2}m-\frac{1}{2}n+1,\ p+\frac{1}{2}m+\frac{1}{2}n+1\ ;\ \frac{1}{2}\right),$$

where $\text{Re}(m)>-1$, $\text{Re}(p)>-\text{Re}(k)-1$, $\text{Re}(p)>\text{Re}(k)$ and

$$(2.9)\quad \int_{0}^{\infty}e^{-pt}(e^{t}-1)^{\frac{1}{2}m}\left(\frac{\rho e^{t}}{\rho-2}-1\right)^{\sigma}P_{k}^{-m,-n}(\rho e^{t}-\rho+1)dt$$

$$=\frac{\rho^{p-\frac{1}{2}m}(\rho-2)^{-\sigma}\Gamma\left(p-\sigma-k-\frac{1}{2}m\right)\Gamma\left(p-\sigma+k-\frac{1}{2}m+1\right)}{\Gamma\left(p-\sigma-\frac{1}{2}n+1\right)\Gamma\left(p-\sigma+\frac{1}{2}n+1\right)}\times$$

$$\times{}_3F_2\left(p-\sigma-k-\frac{1}{2}m,\ p+1,\ p+k-\sigma+1-\frac{1}{2}m\ ;\ p-\sigma-\frac{1}{2}n+1,\ p-\sigma+\frac{1}{2}n+1\ ;\ \frac{2-\rho}{2}\right),$$

where $\text{Re}(\rho)>0$, $\text{Re}(m)>-1$, $\text{Re}(p)>\text{Re}\left(\frac{1}{2}m+\sigma-k\right)-1$, $\text{Re}(p)>\text{Re}\left(\frac{1}{2}m+\sigma+k\right)$.

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