Kyungpook Math. J.
Volume 13, Number 1
June, 1973

SPACES IN WHICH COVERGENT SEQUENCES ARE EVENTUALLY CONSTANT

By Norman Levine

To show the inadaquacy of sequences in topological spaces, the author has often assigned the following problem to his class: Find a set X and two different topologies $\mathscr{T}$ and $\mathscr{U}$ such that for every sequence S in X and for every point x in X, $\lim S=x(\mathscr{T})$ iff $\lim S=x(\mathscr{U})$.

A simple solution is obtained by taking X uncountable, $\mathscr{T}$ discrete and $\mathscr{U}$ the cocountable topology; for in each of the spaces $(X, \mathscr{T})$ and $(X, \mathscr{U})$, a sequence S converges to x iff S is eventually x.

It is the intent of this paper to investigate spaces with precisely this property:

DEFINITION 1. A space $(X, \mathscr{T})$ is an *E-space* iff convergent sequences are eventually constant.

We give two more examples of E-spaces.

EXAMPLE 1. Let $X=[0, 1]$ and $\mathscr{U}=\{U : 0\notin U$ or $0\in U$ and $\mathscr{C}U$ is countable$\}$.

EXAMPLE 2. Let X be the positive integers and let $\mathscr{T}=\{O : 1\notin O$ or $1\in O$ and $\lim N(O;n)/n=1\}$ where $N(O;n)$ is the number of integers in O which are less or equal to n.

We leave it to the reader to verify that the above spaces are E-spaces.

DEFINITION 2. A space $(X, \mathscr{T})$ is called a $T_{1.5}$-space iff every sequence in X has at most one limit.

As the terminology suggests, $T_{1.5}$ is between T_1 and T_2.

We now proceed to characterize E-spaces in

THEOREM 1. *A space* $(X, \mathscr{T})$ *is an E-space iff* (1) $(X, \mathscr{T})$ *is a* $T_{1.5}$*-space and* (2) *every sequentially compact subset of* X *is finite.*

PROOF. Assume that X is an E-space. Let S be a sequence in X for which $\lim S=x$ and $\lim S=y$. There exist integers N and M for which $S(n)=x$ for $n \geq N$ and $S(n)=y$ for $n\geq M$. Thus $x=S(N+M)=y$ and (1) holds.

+

To show (2), let A be an infinite subset of X. It suffices to show that A is not sequentially compact. Take $\{a_n : n\geq 1\}$ an infinite sequence of distinct points in A. Clearly no subsequence of $\{a_n : n\geq 1\}$ can converge since no subsequence can eventually be constant.

Next, assume that (1) and (2) hold and suppose that $(X, \mathcal{T})$ is not an E-space. There exists then a sequence $\{x_n : n\geq 1\}$ and a point x such that $\lim x_n = x$, but $x_n \neq x$ for an infinite number of n : let $A = \{x_n : x_n \neq x\}$. Case 1. A is finite. There exist then x_{n_j} in A and y in A such that $x_{n_j} = y$ for all j. Then $x_{n_j} \to x$ and $x_{n_j} \to y$. But $y \in A$ and $x \notin A$ contradicting (1).

Case 2. A is infinite. We will show that $A \cup \{x\}$ is sequentially compact, contradicting (2). Let $\{y_n : n\geq 1\}$ be any infinite sequence in $A \cup \{x\}$. If $\{y_n : n\geq 1\}$ is a finite set, then clearly there exists a subsequence which converges. So, assume that $\{y_n : n\geq 1\}$ is an infinite set. Choose m_1 such that $y_{m_1} \neq x$. Then $y_{m_1} = x_{n_1}$ for some n_1. Choose $m_2 > m_1$ such that $y_{m_2} \notin \{x, x_1 \cdots, x_{n_1}\}$. Then $y_{m_2} = x_{n_2}$ for some $n_2 > n_1$. Choose $m_3 > m_2$ such that $y_{m_3} \notin \{x, x_1 \cdots, x_{n_2}\}$. Then $y_{m_3} = x_{n_3}$ for some $n_3 > n_2$. Continuing, we have $y_{m_j} = x_{n_j} \to x$ in $A \cup \{x\}$.

COROLLARY 1. *A space $(X, \mathcal{T})$ is an E-space iff* (1) $(X, \mathcal{T})$ *is a T_1-space and* (2) *every infinite sequence of distinct points in X diverges.*

PROOF. Let $(X, \mathcal{T})$ be an E-space. Then (1) above follows from (1) in theorem 1. (2) follows immediately from the definition of an E-space.

Conversely, suppose (1) and (2) hold; suppose further that $(X, \mathcal{T})$ is not an E-space. There exists then a sequence $\{x_n : n\geq 1\}$ in X and a point x in X such that $x_n \neq x$ for an infinite number of n and a point x in X such that $x_n \neq x$ for an infinite number of n and $x_n \to x$. Let $A = \{x_n : x_n \neq x\}$. If A is finite, then $x \in \mathscr{C}A \in \mathcal{T}$ and $x_n \notin \mathscr{C}A$ for an infinite number of n, a contradiction. If A is infinite, take x_{n_j} in A such that $n_1 < n_2 < n_3 \cdots$ and $x_{n_j} \neq x_{n_i}$ when $i \neq j$. Then $\{x_{n_j} : j\geq 1\}$ is an infinite sequence of distinct points which converges (to x) contradicting (2).

COROLLARY 2. *Let $(Y, \mathcal{U})$ be a subspace of $(X, \mathcal{T})$. If $(X, \mathcal{T})$ is an E-space, then $(Y, \mathcal{U})$ is an E-space.*

PROOF. Properties (1) and (2) of corollary 1 are hereditary.

COROLLARY 3. *Let $\mathcal{T} \subset \mathcal{U}$, $\mathcal{T}$ and $\mathcal{U}$ being topologies for X. If $(X, \mathcal{T})$ is an E-space, then $(X, \mathcal{U})$ is an E-space.*

PROOF. Properties (1) and (2) of corollary 1 carry over to larger topologies.

COROLLARY 4. *A space $(X, \mathscr{T})$ is an E-space iff every countable subspace is an E-space.*

PROOF. If $(X, \mathscr{T})$ is an E-space, then every countable subspace is an E-space by corollary 2.

Conversely, suppose $(X, \mathscr{T})$ is not an E-space. There exists then a sequence $\{x_n : n \geq 1\}$ and a point x such that $\lim x_n = x$, but $x_n \neq x$ for an infinite number of n. Let $A = \{x, x_1, x_2, \cdots, x_n, \cdots\}$. Then $(A, A \cap \mathscr{T})$ is a countable subspace of $(X, \mathscr{T})$ which is not an E-space.

THEOREM 2. *Let $(X, \mathscr{T}) = \times \{(X_\alpha, \mathscr{T}_\alpha) : \alpha \in \Delta\}$, all spaces being nonempty. Then $(X, \mathscr{T})$ is an E-space iff* (1) *$(X_\alpha, \mathscr{T}_\alpha)$ is an E-space for every $\alpha \in \Delta$* and (2) *$\{\alpha : X_\alpha$ is not a singleton$\}$ is finite.*

PROOF. Let $(X, \mathscr{T})$ be an E-space. Then (1) follows from corollary 2 and the fact that X_α is homeomorphic to a subspace of X. To show (2), suppose that $\{\alpha : X_\alpha$ is not a singleton$\}$ is infinite. Choose $\{\alpha_i\}$ an infinite sequence of distinct elements such that X_{α_i} is not a singleton. Let A_{α_i} be a two point subset of X_{α_i} for each i and let A_α be a singleton subset of X_α for all $\alpha \neq \alpha_i$. Then $\times \{A_\alpha : \alpha \in \Delta\}$ is a compact metrizable subset of $\times \{X_\alpha : \alpha \in \Delta\}$ and hence is an infinite sequentially compact subset of X contrary to (2) of theorem 1.

Conversely, suppose that (1) and (2) hold above. Let $\{\alpha : X_\alpha$ is not a singleton$\} = \{\alpha_1, \cdots\cdots, \alpha_k\}$ and let $x_n \to x$ in X. Then $x_n(\alpha_i) \to x(\alpha_i)$ in $X\alpha_i$ for $1 \leq i \leq k$. By (1), $x_n(\alpha_i) = x(\alpha_i)$ for $n \geq N_i$ and $x_n(\alpha) = x(\alpha)$ for all n and all $\alpha \neq \alpha_i$. Hence $x_n = x$ for $n \geq N_1 + \cdots + N_k$.

THEOREM 3. *Let $X = \cup \{O_\alpha : \alpha \in \Delta\}$ in a space $(X, \mathscr{T})$ where $O_\alpha \in \mathscr{T}$ for all $\alpha \in \Delta$. Then $(X, \mathscr{T})$ is an E-space iff $(O_\alpha, O_\alpha \cap \mathscr{T})$ is an E-space for each $\alpha \in \Delta$.*

PROOF. If $(X, \mathscr{T})$ is an E-space, then $(O_\alpha, O_\alpha \cap \mathscr{T})$ is an E-space by corollary 2.

Conversely, let $(O_\alpha, O_\alpha \cap \mathscr{T})$ be an E-space for each $\alpha \in \Delta$, and let $x_n \to x$ in X. Then $x \in O_\alpha$ for some $\alpha \in \Delta$ and hence $x_n \in O_\alpha$ for $n \geq N$. Thus $\{x_n : n \geq N\} \to x$ in O_α and since O_α is an E-space, we have $x_n = x$ for $n \geq M \geq N$ for some M.

COROLLARY 5. *Let $(X, \mathscr{T}) = \sum \{X_\alpha, \mathscr{T}_\alpha) : \alpha \in \Delta\}$. Then $(X, \mathscr{T})$ is an E-space iff $(X_\alpha, \mathscr{T}_\alpha)$ is an E-space for each $\alpha \in \Delta$.*

LEMMA 1. *Let $(X, \mathscr{T})$ be a space and suppose that $X = C \cup D$, C and D being closed sets. Then X is an E-space iff C and D are E-spaces in the subspace* *topology.*

PROOF. If X is an E-space, then C and D are E-spaces by corollary 2.

Conversely, suppose that C and D are E-spaces in the relative topology and let $x_n \to x$ in X. Case 1. $x \in C-D$. Then $x \in \mathscr{C}D \in \mathscr{T}$ and hence $x_n \in \mathscr{C}D$ for $n \geq N$ for some N. Thus $\{x_n : n \geq N\}$ is a sequence in C which converges to x and hence $x_n = x$ for $n \geq M \geq N$ for some M. Case 2. $x \in C \cap D$. We consider only the subcase for which $x_n \in C$ for infinitely many n and $x_n \in D$ for infinitely many n. Let $\{x_{n_j}\}$ be the natural subsequence of $\{x_n\}$ determined by C and let $\{x_{m_j}\}$ be the natural subsequence of $\{x_n\}$ determined by D. Then $x_{n_j} = x$ for $j \geq N$ and $x_{m_j} = x$ for $j \geq M$ for some N and M. Thus $x_n = x$ for $n \geq n_N + m_M$.

COROLLARY 6. *Let $X = F_1 \cup \cdots \cup F_n$, $(X, \mathscr{T})$ being a space in which each F_i is closed. Then X is an E-space iff F_i is an E-space in the relative topology for $1 \leq i \leq n$.*

EXAMPLE 3. Let $X = \left\{0,\ 1,\ \frac{1}{2},\ \cdots,\ \frac{1}{n},\ \cdots\right\}$ with the usual topology. If $E_n = \left\{\frac{1}{n}\right\}$ for each n, and $E_0 = \{0\}$, then each E_n is a closed E-space, but $\cup\{E_n : n \geq 0\}$ is not an E-space.

However, we have

THOEREM 4. *Let $\{E_\alpha : \alpha \in \Delta\}$ be a locally finite family of closed sets in a space $(X, \mathscr{T})$ such that $X = \cup\{E_\alpha : \alpha \in \Delta\}$. Then X is an E-space iff E_α is an E-space for each $\alpha \in \Delta$.*

PROOF. If X is an E-space, then E_α is an E-space for each $\alpha \in \Delta$ by corollary 2.

Conversely, let E_α be an E-space for each $\alpha \in \Delta$ and suppose that $x_n \to x$ in X. There exists an open set O such that $x \in O$ and $O \cap E_{\alpha_i} \neq \phi$ for $\alpha_1, \cdots, \alpha_k$ only. Then $O \subset E_{\alpha_1} \cup \cdots \cup E_{\alpha_k}$. There exists an N such that $x_n \in O$ for $n \geq N$. Thus $\{x_n : n \geq N\}$ is an infinite sequence in $E_{\alpha_1} \cup \cdots \cup E_{\alpha_k}$ which converges to x. By corollary 6, $E_{\alpha_1} \cup \cdots \cup E_{\alpha_k}$ is an E-space and hence $\{x_n : n \geq N\}$ is eventually x.

THEOREM 5. *Let $(X, \mathscr{T})$ be a first axiom space. Then X is an E-space iff $(X, \mathscr{T})$ is discrete.*

PROOF. We need only show that if X is an E-space, then $(X, \mathscr{T})$ is discrete. Let $x \in c(A)$ where $A \subset X$ and c is the closure operator. Then there exists a sequence of points $\{a_n\}$ in A such that $a_n \to x$. But $\{a_n : n \geq 1\}$ is eventually x and hence $x \in A$. Thus A is closed.

COROLLARY 7. *If $(X, \mathcal{T})$ is an E-space, then $(X, \mathcal{T})$ is discrete if X is finite or $(X, \mathcal{T})$ is pseudo metrizable.*

DEFINITION 2. For X any set, we denote the cofinite topology on X by $\mathcal{T}_{cf}$.
By (1) of theorem 1, $\mathcal{T}_{cf} \subset \mathcal{T}$ whenever $\mathcal{T}$ is an E-topology on X. If X is infinite, then $\mathcal{T}_{cf}$ is not an E-topology for X; in this case, $\mathcal{T}_{cf}$ is not the largest non-E subtopology of $\mathcal{T}$ as shown in.

THEOREM 6. *Let $(X, \mathcal{T})$ be an infinite E-space. There exists a non-E topology $\mathcal{U}$ on X such that $\mathcal{T}_{cf} \subset \mathcal{U} \subset \mathcal{T}$, the inclusions being proper.*

PROOF. By (1) of theorem 1, $\mathcal{T}_{cf} \subset \mathcal{T}$, the inclusion being proper since $\mathcal{T}_{cf}$ is not an E-topology; take $O \in \mathcal{T} - \mathcal{T}_{cf}$. Then $\mathscr{C}\, O$ is infinite; take $\{x_i : i \geq 1\}$ an infinite sequence of distinct points in $\mathscr{C}O$. Let $\mathcal{U} = \sup\{\mathcal{T}_{cf}, \{\phi, O, X\}\}$. Then $\mathcal{T}_{cf} \subset \mathcal{U}$, $\mathcal{T}_{cf} \neq \mathcal{U}$. Also, $\mathcal{U} \subset \mathcal{T}$. But $\mathcal{U}$ is not an E-space, for $x_i \to x(\mathcal{U})$ for all $x \in \mathscr{C}O$, but $\{x_i : i \geq 1\}$ is not eventually x. Hence $\mathcal{U} \neq \mathcal{T}$.

THEOREM 7. *Let $(X, \mathcal{T})$ be a non E, T_1-space. These exists then a topology $\mathcal{U}$ on X such that $\mathcal{T} \subset \mathcal{U}$, $\mathcal{T} \neq \mathcal{U}$ and $\mathcal{U}$ is not an E-topology for X.*

PROOF. Since $(X, \mathcal{T})$ is not an E-space, there exists an infinite sequence of points $\{x_n : n \geq 1\}$ and a point x such that $x_n \to x(\mathcal{T})$, but $\{x_n : n \geq 1\}$ is not eventually x. Since $(X, \mathcal{T})$ is a T_1-space, we may assume without loss of generality that $x_n \neq x$ for all n and $x_n \neq x_m$ when $n \neq m$. Let $\mathcal{U} = \sup\{\mathcal{T}, \{\phi, \{x, x_2, x_4, x_6, \cdots\}, X\}\}$. Then $\mathcal{U}$ is not an E-topology since $x_{2n} \to x(\mathcal{U})$and $x_{2n} \neq x$ for all n. Furthermore, $\mathcal{T} \subset \mathcal{U}$ and $\mathcal{T} \neq \mathcal{U}$ since $\{x, x_2, x_4, \cdots\} \in \mathcal{U} - \mathcal{T}$.

EXAMPLE 4. Let $X = \{a, b\}$ and $\mathcal{T} = \{\phi, \{a\}, X\}$. Clearly, $\mathcal{T}$ is a non-E-topology for X which is not properly contained in a non-E topology on X.

THEOREM 8. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a continuous surjection and suppose that $\mathcal{T}$ is the weak topology determined by f and $\mathcal{U}$. If $(X, \mathcal{T})$ is an E-space, then $(Y, \mathcal{U})$ is an E-space.

PROOF. Let $y_n \to y$ in Y. There exist x_n in X and x in X such that $f(x_n) = y_n$ and $f(x) = y$. But $x_n \to x$; if $x \in f^{-1}[U]$, then $y = f(x) \in U$ and hence $f(x_n) = y_n \in U$ for $n \geq N$ for some integer N. Thus $x_n \in f^{-1}[U]$ for $n \geq N$. Since $(X, \mathcal{T})$ is an

E-space, $x_n = x$ for $n \geq x$ for $n \geq M$ for some M. Thus $y_n = f(x_n) = f(x) = y$ for $n \geq M$.

EXAMPLE 5. Let $(X, \mathscr{T})$ be an uncountable set with the cocountable topology and let $Y = \{a, b\}$ with $\mathscr{U} = \{\phi, \{a\}, Y\}$. Take $\phi \neq O \neq X$, $O \in \mathscr{T}$: let $f : X \to Y$ as follows: $f(x) = a$ for $x \in O$ and $f(x) = b$ for $x \notin O$. Then f is an identification, $(X, \mathscr{T})$ is an E-space, but $(Y, \mathscr{U})$ is not an *E-space.*

THEOREM 9. *Let $f : (X, \mathscr{T}) \to (Y, \mathscr{U})$ be a continuous injection. If $(Y, \mathscr{U})$ is an E-space, then $(X, \mathscr{T})$ is an E-space.*

The Ohio State University

REFERENCE

John L. Kelley, *General Topology*, Van Nostrand, Princeton, N. J., 1955.