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PAIRWISE LINDELOF BITOPOLOGICAL SPACES

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In a recent paper Fletcher, Hoyle and Patty [1] introduced the concept of pairwise compactness for bitopological spaces. In this note we extend this concept to a larger class of bitopological spaces, called pairwise Lindelof spaces, and prove some results which have well known topological analogues. Bitopological notions not defined here are taken from kelly [2]. Some of the results of this paper were announced in [5].

DEFINITION 1. A cover of a bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is *pairwise open* if its elements are members of $\mathscr{T}_1$ or $\mathscr{T}_2$, and if it contains at least one nonempty member of each of $\mathscr{T}_1$ and $\mathscr{T}_2$.

DEFINITION 2. $(X, \mathscr{T}_1, \mathscr{T}_2)$ is *pairwise Lindelof* (*pairwise compact*) if each pairwise open cover of $(X, \mathscr{T}_1, \mathscr{T}_2)$ has a countable (finite) subcover.

Clearly, any pairwise compact space is pairwise Lindelof.

EXAMPLE 1. Let $X=[0, \Omega]$, $\mathscr{T}_1$ be the discrete topology on X and $\mathscr{T}_2$ be the topology $\{\phi, X, (a, \Omega]$ for each $a\in X\}$. Then $(x, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise Lindelof. If $\mathscr{U}$ is any pairwise open cover of x, there is an $a\in X$ such that $(a, \Omega]\in\mathscr{U}$, and hence $\mathscr{U}$ has a subcover of cardinality not greater than $a+1$. However, $(X, \mathscr{T}_1)$ is uncountable and discrete and hence not Lindelof. $(X, \mathscr{T}_2)$ is Lindelof, indeed compact. Furthermore, $(X, \mathscr{T}_1, \mathscr{T}_2)$ is not pairwise compact.

EXAMPLE 2. Let $x=[0, \Omega)$, $\mathscr{T}_1$ be the ordinal topology on X and $\mathscr{T}_2$ be the topology $\{\phi, X, (0, \Omega), (0, a]$ for each $a\in X\}$. Then $(X, \mathscr{T}_1)$ is not Lindelof, as it is regular but not paracompact, while $(X, \mathscr{T}_2)$ is Lindelof, indeed compact. Moreover, $(X, \mathscr{T}_1, \mathscr{T}_2)$ is not pairwise Lindelof, since any pairwise open cover of X is a $\mathscr{T}_1$ open cover as $\mathscr{T}_2\subset\mathscr{T}_1$.

EXAMPLE 3. In Example 2 change $\mathscr{T}_2$ to the topology $\{\phi, X, (a, \Omega)$ for each $a\in X\}$. Then $\mathscr{T}_2\subset\mathscr{T}_1$, $\mathscr{T}_1$ is not Lindelof, but $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise Lindelof, since any pairwise open cover contains a set of the form (a, Ω) whose complement

$[0, a]$ is $\mathcal{T}_1$ Lindelof.

DEFINITION 3. (Pahk and Choi [4]) A bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is *pairwise countably compact* if every countable pairwise open cover of $(X, \mathcal{T}_1, \mathcal{T}_2)$ has a finite subcover.

The proofs of the next two results are straight forward.

PROPOSITION 1. *In a pairwise Lindelof space, pairwise countable compactness is equivalent to pairwise compactness.*

PROPOSITION 2. *The pairwise continuous image of a pairwise Lindelof space is pairwise Lindelof.*

PROPOSITION 3. *Any second countable bitopological space (that is, a bitopological space in which both topologies are second countable) is pairwise Lindelof.*

PROOF. In the space $(X, \mathcal{T}_1, \mathcal{T}_2)$ let $\{B_n\}_{n=1}^{\infty}$ and $\{C_m\}_{m=1}^{\infty}$ be countable bases for $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively, and $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be a pairwise open cover of X. Let N be the set of integers n such that $B_n \subset U_\alpha$ for some $U_\alpha \in \mathcal{U} \cap \mathcal{T}_1$, and M the set of integers m such that $C_m \subset U_\alpha$ for some $U_\alpha \in \mathcal{U} \cap \mathcal{T}_2$. Denote by V_n one of the U_α in $\mathcal{U} \cap \mathcal{T}_1$ such that $B_n \subset U_\alpha$, and by W_m one of the U_α in $\mathcal{U} \cap \mathcal{T}_2$ such that $C_m \subset U_\alpha$. Then $\mathcal{U}^* = \{V_n : n \in N\} \cup \{W_m : m \in M\}$ is a countable subcover of $\mathcal{U}$ for X. Let $x \in X$. Since $\mathcal{U}$ covers X there is a $U_\beta \in \mathcal{U}$ such that $x \in U_\beta$. Now U_β is either $\mathcal{T}_1$ open or $\mathcal{T}_2$ open. If U_β is $\mathcal{T}_1$ open, there is an integer k such that $x \in B_k \subset U_\beta$, so that $k \in N$. Hence, there is a set $V_k \in \mathcal{U}^*$ such that $x \in B_k \subset V_k$. A similar argument suffices if U_β is $\mathcal{T}_2$ open.

PROPOSITION 4. *If $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise Lindelof and A is a proper subset of X which is $\mathcal{T}_1$ closed then A is pairwise Lindelof and $\mathcal{T}_2$ Lindelof.*

PROOF. Let $\mathcal{U}$ be any pairwise open cover of $(A, \mathcal{T}_1|A, \mathcal{T}_2|A)$. Then $\mathcal{U} \cup \{(X-A)\}$ induces a pairwise open cover of $(X, \mathcal{T}_1, \mathcal{T}_2)$ which has a countable subcover and hence so does $\mathcal{U}$.

Let $\mathcal{W}$ be any $\mathcal{T}_2$ open cover of A. Then $\mathcal{W} \cup \{(X-A)\}$ is a pairwise open cover of $(X, \mathcal{T}_1, \mathcal{T}_2)$ which has a countable subcover, and hence so does $\mathcal{W}$.

THEOREM 1. *If $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise Lindelof and pairwise regular then it is pairwise normal.*

PROOF. Let A be a $\mathcal{T}_1$ closed subset of X and B be a $\mathcal{T}_2$ closed subset of X

disjoint from A. If $A=X$ or $B=X$, the proof is trivial. Otherwise A and B are proper subsets of X. Let x be any point of B. Since $\mathscr{T}_1$ is regular with respect to $\mathscr{T}_2$, there is a $\mathscr{T}_1$ open set U_x such that $x\in U_x$ and $(\mathscr{T}_2 \operatorname{cl} U_x)\cap A=\phi$. (Throughout this paper $\mathscr{T}_2$ cl A denotes the $\mathscr{T}_2$ closure of the set A.) Thus $\mathscr{U}=\{U_x : x\in B\}$ is a $\mathscr{T}_1$ open cover of B which is $\mathscr{T}_2$ closed and hence, by proposition 4, $\mathscr{T}_1$ Lindelof. So there is a countable subcover $\{U_1, U_2, \cdots\}$ of $\mathscr{U}$ for B such that $(\mathscr{T}_2 \operatorname{cl} U_i)\cap A=\phi$ for each positive integer i. By a similar argument, there is a countable $\mathscr{T}_2$ open cover $\{V_1, V_2, \cdots\cdots\}$ for A such that for each positive integer j $(\mathscr{T}_1 \operatorname{cl} V_j)\cap B=\phi$.

If $W_n=V_n-\bigcup_{i\leq n}\{\mathscr{T}_2 \operatorname{cl} U_i\}$ and $Y_m=U_m-\bigcup_{j\leq m}\{\mathscr{T}_1 \operatorname{cl} V_j\}$, then W_n is $\mathscr{T}_2$ open and Y_m is $\mathscr{T}_1$ open, where m and n are arbitrary positive integers. Then $W_n\cap U_j=\phi$ for $j\leq n$, and $Y_j\subset U_j$, so that $W_n\cap Y_j=\phi$ for $j\leq n$. Similarly, $Y_j\cap W_n=\phi$ for $n\leq j$, so that $W_n\cap Y_m=\phi$ for all m and n. Now $\{V_n\}$ is a $\mathscr{T}_2$ open cover for A and no set of the form $\mathscr{T}_2$ cl U_i contains points of A, so that $\{W_n\}$ is a $\mathscr{T}_2$ open cover for A. Thus $A\subset U=\bigcup_n W_n$ and U is $\mathscr{T}_2$ open. Similarly, $B\subset V=\bigcup_m Y_m$ and V is $\mathscr{T}_1$ open. Furthermore, U and V and disjoint. Thus $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise normal.

We have as a corollary a result proved by Kelly [2].

COROLLARY 1. *Any second countable pairwise regular bitopological space is pairwise normal.*

COROLLARY 2. *Any pairwise compact pairwise regular bitopological space is pairwise normal.*

Fletcher, Hoyle and Patty [1] proved the following result.

THEOREM 2. *Any pairwise Hausdorff pairwise compact bitopological space is pairwise regular.*

This, together with Corollary 2, yields the following theorem which is also an immediate consequence of Theorems 12 and 13 of Fletcher, Hoyle and patty [1].

THEOREM 3. *Any pairwise Hausdorff pairwise compact bitopological space is pairwise normal.*

THEOREM 4. *Let $(X, \mathscr{T}_1, \mathscr{T}_2)$ be a bitopological space. If $\mathscr{T}_1$ is regular with respect to $\mathscr{T}_2$ and $(X, \mathscr{T}_1)$ is second countable, then every $\mathscr{T}_1$ closed set is a $\mathscr{T}_2 G_\delta$.*

PROOF. Let A be a $\mathscr{T}_1$ closed set. If $A=X$ we are through. Otherwise, for each $x\notin A$ there is a $\mathscr{T}_1$ open set U_x such that $x\in U_x\subset \mathscr{T}_2 \operatorname{cl} U_x\subset X-A$. Let $\{V_n : n\in N,$

the integers} be a countable base for $(X, \mathscr{T}_1)$. Then $x \in V_{n(x)} \subset U_x$ for some integer $n(x)$. Now $V_{n(x)} \subset \mathscr{T}_2$ cl U_x implies that $(\mathscr{T}_2 \text{ cl } V_{n(x)}) \cap A = \phi$, and hence $A = \bigcap \{X - \mathscr{T}_2 \text{ cl } V_{n(x)} : x \notin A\}$. The number of distinct integers $n(x)$ is countable, so that A is *a* $\mathscr{T}_2$ G_δ.

The next result is an improvement of Corollary 1. First we need the following definition.

DEFINITION 4. (Lane [3]) $(X, \mathscr{T}_1, \mathscr{T}_2)$ is *pairwise perfectly normal* if it is pairwise normal, each $\mathscr{T}_1$ closed set is a $\mathscr{T}_2$ G_δ and each $\mathscr{T}_2$ closed set is a $\mathscr{T}_1$ G_δ.

THEOREM 5. *If* $(X, \mathscr{T}_1, \mathscr{T}_2)$ *is pairwise regular and second countable then it is pairwise perfectly normal.*

PROOF. The pairwise normality of $(X, \mathscr{T}_1, \mathscr{T}_2)$ follows from Corollary 1. Furthermore, closed sets are appropriate G_δs by Theorem 4.

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