A NOTE ON FIBRE BUNDLES

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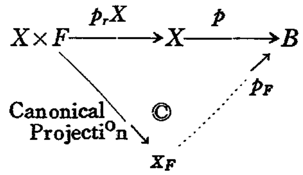
Let $\xi=(X, p, B)$ be a *principal G-bundle* (§2), where G is a topological group ([1], [2]). For a left G-space F the relation $(x, y)s=(xs, s^{-1}y)$ defines a right G-space structure on $X \times F$, where $(x, y) \in X \times F$ and $s \in G$. We put $X_F = X \times F \bmod G$, and we define $p_F : X_F \longrightarrow B$ by the commutative diagram



where $p_rX(x, y)=x$ for all $(x, y) \in X \times F$ (§2).

In this paper, we shall prove that the bundle $\xi[F]=(X_F, p_F, B)$ is a fibre bundle (§1) under some conditions (Theorem 1 of §3).

1. Fibre bundles

Let $\xi=(X, p, B, F)$ is a bundle with fibre F satisfying *local triviality* ([1], [2] and [3]). Thus there is an open covering $\{U_j\}_{j\in J}$ of B such that for each $j \in J$

$$\phi_j : U_j \times F \longrightarrow p^{-1}(U_j)$$

is a homeomorphism. $\{U_j\}_{j\in J}$ is called *a system of coordinate neighborhoods*, and each ϕ_j is called the *coordinate function*. The coordinate functions are required to satisfy the conditions:

$$p\phi_j(b, y)=b \text{ for } (b, y) \in U_j \times F \text{ and for } j \in J.$$

Sometimes, (U_j, ϕ_j) is called a *chart of ξ over U_j*.

Let F be an *effective G-space* (§2), where G is a topological group (that is, G is a group of automorphisms of F) ([1]). We define a map

$$\phi_{j,b} : F \longrightarrow p^{-1}(b)$$

by

$$\phi_{j,b}(y)=\phi_j(b, y)$$

(in the sequel, a *map* means a *continuous map*), then for each pair $(i, j)\in J\times J$ and for $b\in U_i\cap U_j$ G must satisfy the condition that the homeomorphism

$$\phi_{j,b}^{-1}\circ\phi_{i,b} : F\longrightarrow F$$

coincides with the operation of a unique element of G. In this case, the group G is called *structure group of the bundle* ξ.

Thus, the map $g_{j,i} : U_i\cap U_j\longrightarrow G$ defined by

$$g_{j,i}(b)=\phi_{j,b}^{-1}\circ\phi_{i,b}$$

is continuous. We have the following results:

(i) For any $(i, j, k)\in J\times J\times J$

$$g_{k,j}(b)\,g_{j,i}(b)=g_{k,i}(b), \quad b\in U_i\cap U_j\cap U_k.$$

(ii) For $i\in J$ $g_{i,i}(b)=$the identity of $G, b\in U_i$

(iii) In (i), put $i=k$, then from (ii) we obtain

$$g_{j,k}(b)=[g_{k,j}(b)]^{-1}, \quad b\in U_j\cap U_k.$$

If we define the map $p_j : p^{-1}(U_j)\longrightarrow F$ by

$$p_j(x)=\phi_{j,b}^{-1}(x),$$

where $p(x)=b$, the following identities hold.

(iv) $p_j\phi_j(b, y)=y$, $\phi_j(p(x), p_j(x))=x$, $g_{j,i}(p(x))p_i(x)=p_j(x)$,

where $(b, y)\in U_j\times F$, $x\in X$ and $p(x)\in U_i\cap U_j$. $\{g_{i,j}\}_{(i,j)\in J\times J}$ is called a *system of transition functions* of B relative to an open covering $\{U_j\}_{j\in J}$. In this case, for $b\in U_i\cap U_j$, we have

$$\phi_j(b, y)=\phi_i(b, g_{i,j}(b)y).$$

The bundle $\xi=(X, p, B, F, G)$ is called a *coordinate bundle* with charts $\{(U_j, \phi_j)\}$ and the structure group G.

Two coordinate bundles ξ and ξ' are said to be *equivalent in the strict sense* if they have the same bundle space, base space, projection, fibre and groups, and their charts $\{(U_j, \phi_j)\}$, $\{(U'_k, \phi'_k)\}$ satisfy the conditions that

$$\bar{g}_{k,j}(b)=\phi'^{-1}_{k,b}\cdot\phi_{j,b}, \quad b\in U_j\cap U'_k,$$

coincides with the operation of an element of G, and the map

$$\bar{g}_{k,j} : U_j\cap U'_k\longrightarrow G$$

so obtained is continuous.

PROPOSITION 1. *The above relation is a proper equivalence relation.*

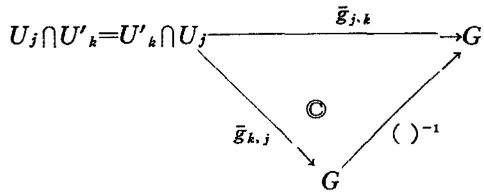
Proof. By definition of $g_{j,i}$, reflexivity is obvious. For

$$\bar{g}_{k,j}(b)=\phi'^{-1}_{k,b}\cdot\phi_{j,b}, \quad b\in U_j\cap U'_k,$$

which is in G,

$$\bar{g}_{j,k}(b)=\phi_{j,b}^{-1}\cdot\phi'_{k,b}$$

is $(\bar{g}_{k,j}(b))^{-1}$ by the above (iii). Therefore we have the commutative diagram



where $(\ )^{-1}: G\longrightarrow G$ is defined by $(\ )^{-1}(g)=g^{-1}$, $g\in G$. Since G is a topological group, $(\ )^{-1}$ is continuous, and therefore $\bar{g}_{j,k}$ is continuous. Symmetry is proved.

Assume that ξ is equivalent in the sense to ξ' and ξ' is equivalent in the sense to ξ''. We want to prove that for all $b\in U_j\cap U''_l$

$$\bar{g}_{l,j}: U_j\cap U''_l\longrightarrow G$$

is continuous at b, where $\bar{g}_{l,j}(b)=\phi''^{-1}_{l,b}\cdot\phi_{j,b}$. Take U'_k such that $b\in U_j\cap U'_k\cap U''_l$, then

$$\bar{g}_{k,j}: U_j\cap U'_k\longrightarrow G,\quad \bar{g}_{l,k}: U'_k\cap U''_l\longrightarrow G$$

are continuous. Since

$$\bar{g}_{l,k}(b)\bar{g}_{k,j}(b)=(\phi''^{-1}_{l,b}\cdot\phi'_{k,b})\cdot(\phi'^{-1}_{k,b}\cdot\phi_{j,b})=\phi''^{-1}_{l,b}\cdot\phi_{j,b}$$

and $G\times G\longrightarrow G$ defined by $(g,g')\longmapsto gg'$ is continuous, $\bar{g}_{l,j}$ is continuous at b. Therefore transitivity is verified. q.e.d.

With this notion of equivalence, a *fibre bundle* is defined to be an equivalence class of coordinate bundles. Thus a fibre bundle may regard a maximal coordinate bundle having all possible coordinate functions of equivalence class ([3], [4]).

2. Principal G-bundles

Let X be a topological space, and let G be a topological group. X is a right G-space if a map $X\times G\longrightarrow X$ defined by $(x,s)\longmapsto xs\in X$ satisfies the following conditions:

(i) For each $x\in X$, $s,t\in G$, the relation $x(st)=(xs)t$ holds.

(ii) For each $x\in X$, the relation $x1=x$ holds, where 1 is the identity of G.

A right G-space X is said to be *effective* if it has the property that $xs=x$ implies $s=1$. Let X^* be the subspace consisting of all $(x,xs)\in X\times X$, where

$x \in X$, $s \in G$ for an effective G-space X. There is a function $\tau : X^* \longrightarrow G$ such that $x\tau(x, xs) = xs$ for all $(x, xs) \in X^*$. The function $\tau : X^* \longrightarrow G$ is called the *translation function*. From the definition of τ we have the following:

(iii) $\tau(x, x) = 1$

(iv) $\tau(x, x')\tau(x', x'') = \tau(x, x'')$

(v) $\tau(x', x) = [\tau(x, x')]^{-1}$ for $x, x', x'' \in X$.

A right G-space X is said to be *principal* if X is a right effective G-space with a continuous translation function $\tau : X^* \longrightarrow G$. A *principal G-bundle* is a G-bundle (X, p, B), where X is principal.

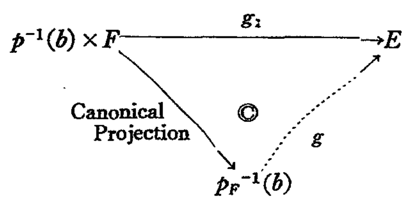
Let $\xi = (X, p, B)$ be a principal G-bundle, and let F be a left G-space. The bundle $\xi[F] = (X_F, p_F, B)$ is called the *associated bundle of ξ with fibre F* (see the first part of this paper). The group G is called the *structure group* of $\xi[F]$.

PROPOSITION 2. *In $\xi[F] = (X_F, p_F, B)$, $p_F^{-1}(b)$ is homeomorphic to F for all $b \in B$.*

Proof. Note that there is the translation map $\tau : X^* \longrightarrow G$ of the G-bundle $\xi = (X, p, B)$. Let $p(x_0) = b$ for some $x_0 \in X$. We define the map $f : F \longrightarrow X_F$ by $f(y) = (x_0, y)G$ for $y \in F$, where $(x_0, y)G$ is an element of X_F. Since $p_F((x_0, y)G) = p(x_0) = b$, $f(F)$ is a subset of $p_F^{-1}(b)$. Define

$$g_1 : p^{-1}(b) \times F \longrightarrow F \text{ by } g_1(x, y) = \tau(x_0, x)y,$$

where $x = x_0 s$ for some $s \in G$. Then $g_1(xs, s^{-1}y) = g_1(x, y)$. If the map $g : p_F^{-1}(b) \longrightarrow F$ is defined by the commutative diagram



we know that f and g are inverse to each other. q.e.d.

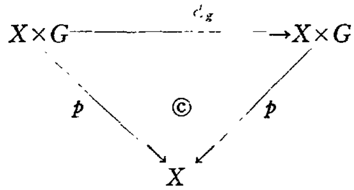
3. The Main Theorem

An *atlas* of charts of a bundle $\xi = (X, p, B, F)$ with fibre F is a family $\{(U_j, \phi_j)\}_{j \in J}$ of charts such that $\bigcup_{j \in J} U_j = B$.

LEMMA 1. *For the product bundle* ([1]) $\xi = (X \times G, p, X)$ *there is a one-to-one correspondence between all X-automorphisms $\xi \longrightarrow \xi$ over X and all maps $X \longrightarrow G$, where G is a topological space. That is, an X-automorphism $\phi_g : \xi$*

$\longrightarrow\xi$ corresponds to the map $g: X\longrightarrow G$ such that $\phi_g(x, s)=(x, g(x)s)$ for $(x, s)\in X\times G$.

Proof. Define $(X\times G)\times G\longrightarrow X\times G$ by $((x, s), t)\longmapsto(x, s)t=(x, st)$. Then $X\times G$ is a right G-space. Since $\phi_g: \xi\longrightarrow\xi$ is an X-automorphism we have the commutative diagram;



where ϕ_g is a G-morphism ([1]). From $p\phi_g=p$, we have $\phi_g(x, s)=(x, f(x, s))$ for some map $f: X\times G\longrightarrow G$. Put $g(b)=f(b, 1)$. Then we have $\phi_g(x, s)=\phi_g(x, 1)s=(x, g(x))s=(x, g(x)s)$.

Conversely, from the relation $\phi_g(x, st)=(x, g(x)s)t=\phi_g(x, s)t$, it follows that ϕ_g is an automorphism with inverse morphism $\phi_g{}^{-1}=\phi_{g'}$, where $g'(x)=g(x)^{-1}$ for $x\in X$.

LEMMA 2. *Let $\xi=(X\times G, p, X)$ be a right G-bundle* ([1]), *and let F be a left effective G-space. The bundle automorphisms*

$$\xi[F]\longrightarrow\xi[F]=(X\times F, q, X)$$

are all of the form $\phi_g(x, y)=(x, g(x)y)$, where $g: X\longrightarrow G$ is a map and $X\times F\approx X\times G\times F \bmod G$.

Proof. By Lemma 1, our bundle automorphisms are quotients of $(x, s, y)\longmapsto(x, g(x)s, y)$. Since $(x, g(x)s, y)=(x, g(x)y) \bmod G$, our bundle automorphisms are of the form $(x, y)\longmapsto(x, g(x)y)=\phi_g(x, y)$. Since $g(x)\in G$, $g(x)$ is an automorphism of F. q.e.d.

PROPOSITION 3. *Let $\xi=(X, p, B)$ be a principal G-bundle, and let $\xi[F]$ be the associated bundle of ξ with fibre F. If (U, ϕ_1) and (U, ϕ_2) are charts of $\xi[F]$ over $U\subset B$, then there is a unique map $g: U\rightarrow G$ such that $\phi_1(b, y)=\phi_2(b, g(b)y)$ for each $(b, y)\in U\times F$, where $g(b)$ is an automorphism of F.*

Proof. By the hypothesis $\phi_2{}^{-1}\cdot\phi_1: U\times F\approx U\times F$. By Lemma 2, the automorphism $\phi_2{}^{-1}\cdot\phi_1$ has the unique map $g: U\longrightarrow G$ such that $\phi_2{}^{-1}\cdot\phi_1(b, y)=(b, g(b)y)$. Since F is a left effective G-space $g(b)\in G$ is an automorphism of F. In this case, we have $\phi_1(b, y)=\phi_2(b, g(b)y)$. q.e.d.

Suppose the associated bundle $\xi[F]=(X_F, p_F, B)$ of a principal G-bundle $\xi=$

(X, p, B). Let $\{(U_j, \phi_j)\}_{j\in J}$ be an atlas of $\xi[F]$. Then $(U_i\cap U_j,\ \phi_i|U_i\cap U_j)$ and $(U_i\cap U_j,\ \phi_j|U_i\cap U_j)$ are charts of $\xi[F]$ over $U_i\cap U_j$. For $b\in U_i\cap U_j$ we define

$$\begin{array}{ccc} \phi_{i,b} : F & \longrightarrow & p^{-1}{}_F(b) \\ \Cup & & \Cup \\ y & \longmapsto & \phi_i(b, y) \end{array}$$

and

$$\begin{array}{ccc} \phi_{j,b} : F & \longrightarrow & p^{-1}{}_F(b) \\ \Cup & & \Cup \\ y & \longmapsto & \phi_j(b, y) \end{array}$$

Then, by Proposition 3 there is a unique map

$$g_{j,i} : U_i\cap U_j \longrightarrow G$$

such that $g_{j,i}(b)=\phi^{-1}{}_{j,b}\cdot\phi_{i,b}$, where $g_{j,i}(b)$ is an automorphism of F. If all $g_{j,i}(b)$ for $b\in U_i\cap U_j$ and $(i, j)\in J\times J$ are in G, then $\{g_{j,i}\}_{(i,j)\in J\times J}$ is a system of transition functions of B relative to $\{U_j\}_{j\in J}$, because of we can easily prove that $g_{j,i}$ satisfies the properties (i)-(iii) in §1.

THEOREM 1. (Main theorem) *Let $\xi[F]$ be the associated bundle with fibre F of a principal G-bundle $\xi=(X, p, B)$. If $\xi[F]$ has an atlas $\{(U_j, \phi_j)\}_{i\in J}$, then $\xi[F]$ is a fibre bundle.*

Proof. We have already proved that $\xi[F]$ is a coordinate bundle. Therefore we have to show that two coordinate bundles $\xi[F]$ with atlas $\{(U_j, \phi_j)\}$ and $\xi[F]$ with atlas $\{(U'_j, \phi'_j)\}$ relative to G are equivalent in the strict sense. Since $(U_j\cap U'_i,\ \phi_j|U_j\cap U'_i)$ and $(U_j\cap U'_i,\ \phi'_j|U_j\cap U'_i)$ are charts of $\xi[F]$ over $U_j\cap U'_i$, by Proposition 3 $\{\xi[F], \{(U_j, \phi_j)\}\}$ and $\{\xi[F], \{(U'_j, \phi'_j)\}\}$ are equivalent in the strict sense (§1). q.e.d.

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