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STRUCTURE OF COUPLE CATEGORIES

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1. Introduction

Let **S** be the category of all sets and **C** be a category. A covariant functor $F: \mathbf{C} \to \mathbf{S}$ will be called a *grounding* of **C**. Dually, a contravariant functor $F: \mathbf{C}^* \longrightarrow \mathbf{S}$ is called a cogrounding of **C**, where $\mathbf{C}^*$ is the dual category of **C**. Cat(**C**, **S**) denotes the functor category, which has functors from a small category **C** to **S** for objects and for morphisms the natural transformations between two functors from **C** to **S**, and Co(**C**, **S**) denotes the couple category of **C** [5].

It is known that any small category **C** can be embedded into Cat(**C**, **S**) and Co(**C**, **S**)

In this note, we shall be to establish the results on the conjugates of groundings and cogroundings of a category **C** and some results on couple categories. Next we shall prove that the functor category Cat($\mathbf{C}^*$, **S**) and the full subcategory Con(**C**, **S**) of Co(**C**, **S**), has the natural couple (G, G^*, m) for objects and for morphisms the conjoint transformations $(G, G^*, m) \longrightarrow (G', G'^*, m)$, are equivalent.

2. Conjugates of groundings and cogroundings

Any cogrounding F of a category **C** is said to be *dominated* by a set S of objects of **C** if every set $F(A)$, A is an objects in **C**, is a union of sets $F(f)$ $[F(B)]$, B ranging over the elements of S and f ranging over morphisms in $\mathrm{Hom}_{\mathbf{C}}(A, B)$.

If F is dominated by some set of objects, it is called *proper*.

DEFINITION 1. Let $F: \mathbf{C}^* \longrightarrow \mathbf{S}$ be a cogrounding of a category **C**. The grounding $F^*: \mathbf{C} \longrightarrow \mathbf{S}$, such that for each object A of **C**, $F^*(A)$ is the set Hom (F, h_A) of all natural transformations from F to $h_A = \mathrm{Hom}_{\mathbf{C}}(\ , A)$ and for each morphism $f: A \longrightarrow A'$ in **C**, $p \in F(W)$ and $\phi \in F^*(A)$,

$$[F^*(f)(\phi)]_W(p) = f \cdot \phi_W(p)$$

is called the *conjugate* of F.

Dually we can define the conjugate G_* of grounding G as follows;

$$G_*(A)=\text{Hom}\ (G, h^A),$$
$$[G_*(f)(\xi)]_W(p)=\xi_W(p)\cdot f$$

for $p\in G(W)$, $f; A\longrightarrow B$, $\xi\in G_*(B)$.

PROPOSITION 1. *If a cogrounding F of $\mathbf{C}$ and its conjugate F^* are proper then there is a natural transformation from F to $F_*{}^*$.*

Proof. Let $\eta : F\longrightarrow F_*{}^*$ be as follows: For each $A\in\text{Ob}(\mathbf{C})$ (=the class of objects of $\mathbf{C}$),

$$\begin{array}{rcl}\eta_A : F(A) & \longrightarrow & F_*{}^*(A)=\text{Hom}(F^*,\ h_A)\\ \Cup & & \Cup \\ p & \longmapsto & \hat{p}\end{array}$$

where $p_B(p^*)=p^*{}_A\ (p)$ for all $p^*\in F^*(B)$. Then for $g : A\longrightarrow A'$ in $\mathbf{C}$ and $p'\in F(A')$,

$$[\eta_A\cdot F(g)]\ (p')=\eta_A[F(g)(p)]$$
$$[F_*{}^*(g)\cdot\eta_{A'}](p')=F_*{}^*(g)\ [\eta_{A'}(p')]=h^{(g)}\cdot\eta_{A'}(p'),$$

where $h^{(g)} : h^{A'}\longrightarrow h^A$.

For $B\in\text{Ob}(\mathbf{C})$ and $q\in F^*(B)$

$$[\eta_A[F(g)(p')]]_B(q)=q_A[F(g)(p')]$$

while

$$[h^{(g)}\cdot\eta_A(p')]_B(q)=h_B{}^{(g)}\ [\eta_{A'}(p')]_B\ (q)$$
$$=q_{A'}(p')\cdot g.$$

Since $q : F\longrightarrow h_B$ is a natural transformation, we have

$$q_A[F_{(g)}(p')]=h^{(g)}[q_{A'}(p')]=q_{A'}(p')\cdot g$$

Therefore $\eta : F\longrightarrow F_*{}^*$ is a natural transformation.

We say that a grounding (cogrounding) F of $\mathbf{C}$ is *reflexive* if F *is* proper, and F^* is proper and the natural transormation $\eta : F\longrightarrow F_*{}^*$ is the natural equivalence.

PROPOSITION 2. *Let* $\Gamma_1 : \text{Cat}(\mathbf{C}^*, \mathbf{S})\longrightarrow\text{Cat}(\mathbf{C}, \mathbf{S})$ *and* $\Gamma_2 : \text{Cat}(\mathbf{C}, \mathbf{S})^*\longrightarrow \text{Cat}(\mathbf{C}^*, \mathbf{S})$ *be two contravariant functors such that* $\Gamma_1(F)=F^*$ *for all* $F\in \text{Ob}(\text{Cat}(\mathbf{C}^*, \mathbf{S}))$ *and for* $\eta : F_1\longrightarrow F_2$ *in* $\text{Cat}(\mathbf{C}^*, \mathbf{S})$, $\xi\in F_2{}^*(X)$, $[\Gamma_1(\eta)]_X(\xi)=\xi\cdot\eta$, *and for* $G\in\text{Ob}(\text{Cat}(\mathbf{C}, \mathbf{S}))$, $\mu : G_1\longrightarrow G_2$ *in* $\text{Cat}\ (\mathbf{C}, \mathbf{S})$, $\phi\in G_{2*}(X)$, $\Gamma_2(G)=G_*$, $[\Gamma_2(\mu)]_X(\phi)=\phi\cdot\mu$. *Then the two functors* $\text{Hom}_{\text{Cat}(\mathbf{C}^*, \mathbf{S})}(-,\ \Gamma_2)$ *and* $\text{Hom}_{\text{Cat}(\mathbf{C}, \mathbf{S})}(-,\ \Gamma_1)$ *from the product category* $\text{Cat}(\mathbf{C}^*, \mathbf{S})^*\times\text{Cat}(\mathbf{C}, \mathbf{S})^*$ *to* $\mathbf{S}$ *are naturally equivalent, where for* $(F, G)\in\text{Cat}(\mathbf{C}^*,\ \mathbf{S})^*\times\text{Cat}(\mathbf{C},\ \mathbf{S})^*$, [Hom $(-, \Gamma_1-)$] $(F, G)=\mathrm{Hom}(G, \Gamma_1(F)=F^*)$ *and* [Hom$(-, \Gamma_2-)$] (F, G) $=\mathrm{Hom}\ (F, \Gamma_2(G)=G_*)$ *respectively.*

Proof. For each $(F, G)\in \mathrm{Cat}(\mathbf{C}^*, \mathbf{S})^*\times \mathrm{Cat}(\mathbf{C}, \mathbf{S})^*$, we define a mapping $\psi_{(F,G)}: \mathrm{Hom}\ (G, F^*)\longrightarrow \mathrm{Hom}(F, G_*)$ as follows; for $\phi\in \mathrm{Hom}(G, F^*)$, $p\in F(X)$, $q\in G(W)$ and $X, W\in \mathrm{Ob}(\mathbf{C})$

$$[[\psi_{(F,G)}\ (\phi)]_X(p)]_W(q)=[\phi_W(q)]_X(p).$$

Then for $\phi_1, \phi_2\in \mathrm{Hom}(G, F^*)$, if

$$\varphi_{(F,G)}(\phi_1)=\psi_{(F,G)}(\phi_2),$$

$$[[\phi_1]_W(q)]_X(p)=[[\phi_2]_W(q)]_X(p)$$

for all $p\in F(X)$ and $q\in G(W)$. We have $[\phi_1]_W=[\phi_2]_W$ and $\phi_1=\phi_2$. Next we shall show that the diagram

$$\begin{array}{ccc} \mathrm{Hom}(G_1, F_1^*) & \xrightarrow{\psi_{(F_1,G_1)}} & \mathrm{Hom}(F_1, G_1^*) \\ \mathrm{Hom}(\eta, \Gamma_1(\nu)) \downarrow & & \downarrow \mathrm{Hom}(\nu, \Gamma_2(\eta)) \\ \mathrm{Hom}(G_2, F_{2*}) & \xrightarrow{\psi_{(F_2G_2)}} & \mathrm{Hom}(F_2, G_{2*}) \end{array}$$

where $\eta: G_2\longrightarrow G_1\in \mathrm{Cat}\ (\mathbf{C}, \mathbf{S})$ and $\nu: F_2\longrightarrow F_1\in \mathrm{Cat}\ (\mathbf{C}^*, \mathbf{S})$, is commutative. For each $\phi\in \mathrm{Hom}(G_1, F_1^*)$,

$$[\mathrm{Hom}(\nu, \Gamma_2(\eta))\cdot\psi_{(F_1,G_1)}](\phi)=\Gamma_2(\eta)\cdot\psi_{(F_1,G_1)}(\phi)\cdot\nu$$

and for $p_2\in F_2(X)$, $q_2\in G_2(W)$,

$$\begin{aligned}&[[\Gamma_2(\eta)\cdot\psi_{(F_1,G_1)}(\phi)\cdot\nu]_X(p_2)]_W(q_2)\\&=[[\Gamma_2(\eta)]_X\cdot[\psi_{(F_1,G_1)}(\phi)]_X\cdot\nu_X(p_2)]_W(q_2)\\&=[[\psi_{(F_1,G_1)}(\phi)]_X\ \nu_X(p_2)]_W\cdot[\eta_W(q_2)]\\&=[\phi_W[\eta_W(q_2)]]_X\cdot(\nu_X(p_2)).\end{aligned}$$

While

$$\begin{aligned}&[[[\psi_{(F_2,G_2)}\cdot\mathrm{Hom}(\eta,\ \Gamma_1(\nu))](\phi)]_X(p_2)]_W(q_2)\\&=[[\psi_{(F_2,G_2)}\cdot(\Gamma_1(\nu)\cdot\phi\cdot\eta)]_X(p_2)]_W(q_2)\\&=[(\Gamma_1(\nu)\cdot\phi\cdot\eta)_W(q_2)]_X(p_2)\\&=[[\Gamma_1(\nu)]_W\cdot\phi_W\cdot\eta_W(q_2)]_X(p_2)\\&=[\phi_W\cdot\eta_W(q_2)]_X(\nu_X(p_2)).\end{aligned}$$

Therefore ψ is a natural equivalence.

A *diagram* in a category $\mathbf{C}$ is a functor $D: \mathbf{I}\longrightarrow\mathbf{C}$ whose domain category $\mathbf{I}$ is a small category, [2], [3].

DEFINITION 2. The *limit functor* $\wedge_D$ of a diagram D in a category $\mathbf{C}$ is the functor

$$\begin{array}{ccccc} \wedge_D = h_D\cdot K : \mathbf{C} & \longrightarrow & \mathrm{Cat}(\mathbf{I}, \mathbf{C}) & \longrightarrow & \mathbf{S} \\ \Cup & & \Cup & & \Cup \\ W & \longmapsto & K_W & \longmapsto & \mathrm{Hom}(K_W, D) \end{array}$$

where for all $W\in \mathrm{Ob}(\mathbf{C})$, K_W is the constant functor, and a *limit* of the diagram D is a representation (W_0, ϕ) of $\wedge_D$ (cf. [6]), where ϕ is a natural equivalence from h_W to $\wedge_D$. Dually we can define the colimit functor and colimit of a diagram in $\mathbf{C}$.

PROPOSITION 3. *There exists a natural equivalence from the limit functor of a diagram D in a category $\mathbf{C}$ to the conjugate of a grounding of $\mathbf{C}$.*

Proof. Let $D:\mathbf{I}\longrightarrow\mathbf{C}$ be a diagram in $\mathbf{C}$ and $\wedge_D=h_D\cdot K$ be the limit functor of D. If we define $G_D:\mathbf{C}\longrightarrow\mathbf{S}$ as follows; $G_D(X)$ is a disjoint union $\bigcup_{i\in I} h^{D(i)}(X)$ of the sets $h^{D(i)}(X)$ for $X\in\mathrm{Ob}(\mathbf{C})$, and $[G_D(p)](f)=p\cdot f$ for $p: X\longrightarrow Y$ in $\mathbf{C}$, $f\in G_D(X)$, then G_D is a grounding of $\mathbf{C}$. Let define the mapping $\psi_W: \Lambda_0(W)\longrightarrow G_{D*}(W)$ for $W\in \mathrm{ob}(\mathbf{C})$ and $\varphi\in\Lambda_D(W)$ as follows;

$$[\psi_W(\varphi)]_X(x)=x\cdot\varphi\cdot(i)$$

where $x\in h^{D(i)}(X)\subset G_D(X)$ and $i\in\mathbf{I}$. On the other hand if we define the mapping $\phi_0(i): K_W(i)\longrightarrow D(i)$ as follows; $\phi_0(i)=[\phi_{D(i)}][1_{D(i)})$ for an element $\phi\in G_{D*(W)}$ and $i\in\mathbf{I}$, then since ϕ is a natural transformation, $h^W(D(m))\cdot\phi_{D(i)}=\phi_{D(i)}\cdot G_D(D(m))$ for $m: i\longrightarrow j$ in $\mathbf{I}$ Hence for $1_{D(i)}: D(i)\longrightarrow D(i)$ (*in* $\mathbf{C}$),

$$\begin{aligned}[][h^W(D(m))\cdot\phi_{D(i)}](1_{D(i)}) &= h^W(D(m))[\phi_{D(i)}(1_{D(i)})] \\ &= h^W((D(m))\phi_0(i)=D(m)\phi_0(i)\end{aligned}$$

$$[\phi_{D(i)}\cdot G_D(D(m))](1_{D(i)})=\phi_{D(j)}(1_{D(j)})=\phi_0(j),$$

that is, the diagram

$$\begin{array}{ccc} K_W(i)=W & \longrightarrow & D(i) \\ \Big\| & & \Big\downarrow D(m) \\ K_W(j)=W & \longrightarrow & D(j) \end{array}$$

commutes. Therefore ϕ_0 is a natural transformation from K_W to D and if we take $[\psi_W(\phi_0)]=\phi$, then ψ_W is an isomorphism. For a morphism $\alpha: V\longrightarrow W\in\mathbf{C}$, consider the diagram

$$\begin{array}{ccc} L_D(W) & \xrightarrow{\psi_W} & G^*(W) \\ L_D(\alpha)\Big\downarrow & & \Big\downarrow G_{D*}(\alpha) \\ L_D(V) & \xrightarrow[\psi_V]{} & G_*(V). \end{array} \qquad (1)$$

For $x \in (\mathrm{Hom}(D(j), X))$ $(\subset G_D(x))$ and $\varphi \in L_D(W)$

$$[[G_{D*}(\alpha)\psi_W](\varphi)]_X(x)$$
$$=[G_{D*}(\alpha)[\psi_W(\varphi)]_X(x)$$
$$=[h^{(\alpha)}\cdot(\psi)]_X(x)$$
$$=h_X^{(\alpha)}[\psi_W(\varphi)]_X(x)$$
$$=h_X^{(\alpha)}(x\cdot\varphi(j))=x\cdot\varphi(j)\cdot\alpha, \qquad (2)$$

while

$$[[\psi_V\cdot L_D^{(\alpha)}](\varphi)]_X(x)$$
$$=[\psi_V\cdot(\varphi\cdot K(\alpha)]_X(x)$$
$$=x\cdot[(\varphi\cdot K(\alpha))(j)]$$
$$=x\cdot[\varphi(j)\cdot K(\alpha)(j)]=x\cdot\varphi(j)\cdot\alpha \qquad (3)$$

From (2) and (3) we prove that the diagram (1) is commutative. Hence we have the natural equivalence $\psi : L_D \cong G_{D*}$. This completes the proof.

3. Couple categories

This section of the paper is a sequel to my paper on couple categary [5].

DEFINITION 3. A *coupling* of a cogrounding F and a grounding G of a category $\mathbf{C}$ is a function.

$$m : \bigcup_{(X,Y)\in \mathbf{C}\times\mathbf{C}} F(X)\times G(Y) \longrightarrow \bigcup_{(X,Y)\in\mathbf{C}\times\mathbf{C}} \mathrm{Hom}_{\mathbf{C}}(X, Y)$$

such that

$$\begin{array}{ccc} F(X)\times G(Y) & \longrightarrow & \mathrm{Hom}_{\mathbf{C}}(X, Y) \\ \Cup & & \Cup \\ (p, q) & \longmapsto & m(p, q) \end{array}$$

and for $f : W \longrightarrow X$ and $g : Y \longrightarrow Z$ in $\mathbf{C}$,

$$m(F(f)(p),\ G(g)(q))=g\cdot m(p\cdot q)\cdot f.$$

PROPOSITION 4. *There is a one-to-one corespondence between couplings m of F and G, and natural transformation $\mu : G \longrightarrow F^*$ defined by*

$$[\mu_Y(q)]_X(p)=m(p, q) \qquad (4)$$

for $p\in F(X)$, $q\in G(Y)$.

Proof. Let K be a set of all couplings of F, G, and $\psi : K \longrightarrow \mathrm{Hom}(G, F^*)$ such that $\psi(m)=\mu$ where μ satisfies the condition (4). Then $\psi(m)=\psi(n)$ implies $m=n$. For any $\mathrm{Hom}(G, F^*)$, let m be a function from $\bigcup F(X)\times G(Y)$ to $\bigcup \mathrm{Hom}(X, Y)$ satisfying the condition (4). Then for $f : W \longrightarrow X$ and $g : Y \longrightarrow Z$

$$m(F(f)(p),\ G(g)(q)(=[\mu_Z(G(g)(q)]_W(F(f)(p))$$

$$=[h_g\cdot\mu_Y(q)]_W(F(f)(p))$$
$$=g[[\mu_Y(q)]_W(F(f)(p)]$$
$$=g[h_Y(f)\cdot[\mu_Y(q)]_X(p)]$$
$$=g[[\mu_Y(q)]_X(p)\cdot f$$
$$=g\cdot m(p\cdot q)\cdot f$$

Hence m is a coupling of F, G. This completes the proof.

A *grounding couple* on a category **C** is a triple $F=('F, F', m_F)$ consisting of a cogrounding $'F$, grounding F' of **C** and coupling m_F of $'F, F'$.

Morphisms $F=('F, F', m_F)\longrightarrow G=('G, G', m_G)$ of grounding couples are *conjoint transformations* $\eta=('\eta, \eta')$ which are ordered pair of natural transformations $'\eta : 'F\to 'G$, $\eta' : G'\to F'$, satisfying $m_G(\eta'_X(p), q)=m_F(p, \eta'(q))$ for $p\in 'F(X)$, $q\in G'(Y)$.

The category, which has these grounding couples for objects and for morphisms the conjoint transformations, is called a *couple category* of **C** and we denote it by $\mathrm{Co}(\mathbf{C}, \mathbf{S})$ [5].

Let G be a cogrounding of a category **C** and a mapping

$$m : \bigcup_{(X,Y)\in C\times C} G(X)\times G^*(Y)\longrightarrow \bigcap_{(X,Y)\in C\times C} \mathrm{Hom}_C(X, Y)$$

satisfy $m(p, q)=q_X(p)$ for $p\in G(X)$ and $q\in G^*(Y)$. Then for $f : W\longrightarrow X$ and $g : Y\longrightarrow Z$ in **C**, we have

$$m(G(f)(p),\ G^*(g)(q))=[G^*(g)(q)]_W(G^*(f)(p))$$
$$=[h^{(g)}(q)]_W((G(f)(p))$$
$$=h_W{}^{(g)}[q_W(G(f)(p))]$$
$$=q_W(G(f)(p))$$
$$=h_W{}^{(g)}q_X(p)$$
$$=g\cdot q_X(p)\cdot f$$
$$=g\cdot m(p\cdot q)\cdot f$$

Hence m_G is a coupling of G and G^*. A couple (G, G^*, m_G) $(\in\mathrm{Co}(\mathbf{C}, \mathbf{S}))$ is called a *natural couple*. For two natural couples (G, G^*, m_G) and $(G', G'^*, m_{G'})$ let $\eta : G\longrightarrow G'$ and $\eta^* : G'^*\longrightarrow G^*$ be natural transformations such that $\eta^*{}_Y(q)=q\cdot\eta$ for $q\in G'^*(Y)$. Then for $p\in G(x)$ and $q\in G'^*(Y)$

$$m(p, \eta^*{}_Y(q))=[\eta^*{}_Y(g)]_X(p)$$
$$=[q\cdot\eta]_X(p)$$
$$=q_X\eta_X(p)$$
$$=m(\eta_X(p), q).$$

Hence (η, η^*) is conjonint transformation from (G, G^*, m_G) to $(G', G'^*, m_{G'})$ Thus we have the full subcategory of $\mathrm{Co}(\mathbf{C}, \mathbf{S})$, which has natural couples for objects and for morphisms conjoint tranformations (η, η^*). The full subcategory of natural couples is called the *natural couple category* and denoted by $\mathrm{Con}(\mathbf{C}, \mathbf{S})$.

LEMMA. *A functor* $T : \mathbf{A} \longrightarrow \mathbf{B}$ *is an equivalence if and only if there is a functor* $S : \mathbf{B} \longrightarrow \mathbf{A}$ *together with natural equivalences*

$$\varphi : 1_B \cong T \cdot S \text{ and } \psi : ST \cong 1_A, \ [4]$$

PROPOSITION 5. *The functor* $\psi : \mathrm{Cat}(\mathbf{C}^*, \mathbf{S}) \longrightarrow \mathrm{Con}(\mathbf{C}, \mathbf{S})$ *such that* $\psi(G) = (G, G^*, m_G)$ *for all* $G \in \mathrm{Ob}(\mathrm{Cat}(\mathbf{C}^*, \mathbf{S}))$ *and* $\psi(\eta) = (\eta, \eta^*)$ *for all morphisms* η *in* Cat $(\mathbf{C}^*, \mathbf{S})$ *is an equivalence functor.*

Proof. Let $\phi : \mathrm{Con}(\mathbf{C}, \mathbf{S}) \longrightarrow \mathrm{Cat}(\mathbf{C}^*, \mathbf{S})$ be a functor such that $\phi(G, G^*, m_G) = G$ and $\phi(\eta, \eta^*) = \eta$ for $(\eta, \eta^*) : (G, G^*, m_G) \longrightarrow (G', G'^*, m_{G'})$ and $\varphi : 1_{\mathrm{Cat}(\mathbf{C}^*, \mathbf{S})} \longrightarrow \phi \cdot \psi$. Then $\varphi_G : 1_{\mathrm{Cat}(\mathbf{C}^*, \mathbf{S})}(G) \longrightarrow \phi \cdot \psi(G) = G$ is an identity morphism and φ is a natural equivalence. Similarly we can obtain the natural equivalence $\varphi' : \psi\phi \sim 1_{\mathrm{Con}(\mathbf{C}^*, \mathbf{S})}$. By the lemma the functor ψ is an equivalence.

COROLLARY 1. *The functor category* Cat $(\mathbf{C}^*, \mathbf{S})$ *is embedded into the couple category* $\mathrm{Co}(\mathbf{C}, \mathbf{S})$ *and* $\mathbf{C}$ *is embedded into* $\mathrm{Con}(\mathbf{C}, \mathbf{S})$.

A grounding couple $F = ('F, F', m_F)$ is said to be *separated* if $\mu' : F' \longrightarrow 'F^*$ and $'\mu : 'F \longrightarrow F_*'$ are monomorphic ([5], [3]). A cogrounding G is said to be *separated* if its natural couple (G, G^*, m_F) is separated.

DEFINITION 4. A subcategory $\mathbf{A}$ of a category $\mathbf{B}$ is called a *separating subcategory* if for every two distinct morphisms $f, g : X \longrightarrow Y$ in $\mathbf{B}$, there exist an object Z in $\mathbf{A}$ and $h : Y \longrightarrow Z$ such that $h \cdot f \neq h \cdot g$.

We can know that a cogrounding F of $\mathbf{C}$ is separated iff F and F^* are reflexive. By Proposition 5, we have the

COROLLARY 2. *The natural couple category* $\mathrm{Con}(\mathbf{C}, \mathbf{S})$ *is separated iff* $\mathrm{Cat}(\mathbf{C}, \mathbf{S})$ *is separated.*

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