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ON EXTENDED PROJECTIVE CHANGE OF CONNECTIONS

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1. Introduction

Let M be an n-dimensional differentiable manifold in which a system of paths is given by

$$\frac{d^2x^h}{dt^2}+\Gamma_{jk}{}^h(x)\frac{dx^j}{dt}\frac{dx^k}{dt}=\lambda\frac{dx^h}{dt}.$$

A change of $\Gamma_{ji}{}^h(=\Gamma_{ij}{}^h)$ which does not change the system of paths is given by

$$\bar{\Gamma}_{ji}{}^h=\Gamma_{ji}{}^h+\delta_j{}^h p_i+\delta_i{}^h p_j,$$

where p_i is an arbitrary covector field, and is called a projective change of Γ.

Let $\mathfrak{M}$ be a Kählerian manifold with Hermitian metric tensor g_{ji} and complex structure tensor $F_i{}^h$. A change of Hermitian metric is given by

$$\bar{g}_{ji}=e^{2p}g_{ji},\quad \bar{F}_i{}^h=F_i{}^h,\quad \bar{F}_{ji}=e^{2p}F_{ji},$$

where p is a scalar function. This induces a change of connection given by

$$\bar{\Gamma}_{ji}{}^h=\Gamma_{ji}{}^h+\delta_j{}^h p_i+\delta_i{}^h p_j-g_{ji}p^h+F_j{}^h q_i+F_i{}^h q_j-F_{ji}q^h$$

where

$$p_i=\partial_i p,\quad p^h=p_t g^{th},\quad q_i=-p_t F_i{}^t,\quad q^h=q_t g^{th} \text{ and } \bar{\Gamma}_{ji}{}^h-\bar{\Gamma}_{ij}{}^h=2F_{ij}q^h.$$

We call such an affine connection a complex conformal connection. This definition is primarily given by Yano in [1].

2. Extension of projective change of connections

We consider an n-dimensional differentiable manifold M, in which two different connections $\Gamma_{ji}{}^h$, $\bar{\Gamma}_{ji}{}^h$ are given.

Let v^h be a parallel vector field with respect to $\Gamma_{ji}{}^h$ along the path $x^h(t)$ is given by

$$\frac{d^2x^h}{dt^2}+\bar{\Gamma}_{jk}{}^h\frac{dx^j}{dt}\frac{dx^k}{dt}=\varphi_1(x)\frac{dx^h}{dt}, \tag{2.1}$$

then v^h satisfies following equations

$$\frac{dv^h}{dt}+\Gamma_{jk}{}^h\frac{dx^j}{dt}v^k=\varphi_2(t)v^h. \tag{2.2}$$

If there exists a mixed tensor $H_i{}^h$ which is related by the equation

$$v^h=H_j{}^h\frac{dx^j}{dt} \tag{2.3}$$

along the path $x^h(t)$, then we shall call such a change of connection a *pseudo projective change related to H.*

THEOREM 1. *A pseudo projective change related to H of an affine connection, in general, is given by*

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$$\frac{1}{2}(\bar{\Gamma}_{jk}{}^h+\bar{\Gamma}_{kj}{}^h)=\frac{1}{2}(\Gamma_{jk}{}^h+\Gamma_{kj}{}^h)+u_j\delta_k{}^h+u_k\delta_j{}^h+\frac{1}{2}\bar{H}_l{}^h(\nabla_jH_k{}^l+\nabla_kH_j{}^l), \tag{2.4}$$

where u_j *is an arbitrary covector,* $\bar{H}_t{}^hH_k{}^t=\delta_k{}^h$ and ∇_j is covariant derivative with respect to $\Gamma_{jk}{}^h$.

Proof. From (2.1) and (2.2), we can express

$$\frac{dx^i}{dt}\left(\frac{d^2x^h}{dt^2}+\bar{\Gamma}_{jk}{}^h\frac{dx^j}{dt}\frac{dx^k}{dt}\right)-\frac{dx^h}{dt}\left(\frac{d^2x^i}{dt^2}+\bar{\Gamma}_{jk}{}^i\frac{dx^j}{dt}\frac{dx^k}{dt}\right)=0, \tag{2.5}$$

$$v^i\left(\frac{dv^h}{dt}+\Gamma_{jk}{}^h\frac{dx^j}{dt}v^k\right)-v^h\left(\frac{dv^i}{dt}+\Gamma_{jk}{}^i\frac{dx^j}{dt}v^k\right)=0. \tag{2.6}$$

Differentiating (2.3), we have

$$\frac{dv^h}{dt}=\partial_kH_j{}^h\frac{dx^j}{dt}\frac{dx^k}{dt}+H_j{}^h\frac{d^2x^j}{dt^2}. \tag{2.7}$$

Substituting (2.7) in (2.6), and eliminating $\frac{d^2x^h}{dt^2}$ from (2.5) and (2.6), we can have

$$\{H_t{}^iH_l{}^h\bar{\Gamma}_{jk}{}^t-H_l{}^iH_t{}^h\bar{\Gamma}_{jk}{}^t+H_l{}^tH_k{}^i\Gamma_{jt}{}^h-H_l{}^tH_k{}^h\Gamma_{jt}{}^i$$
$$+H_l{}^i(\partial_kH_j{}^h)-H_l{}^h(\partial_kH_j{}^i)\}\frac{dx^j}{dt}\frac{dx^k}{dt}\frac{dx^l}{dt}=0.$$

By assumption, the last equation is valid for any tangent vector $\frac{dx^h}{dt}$ along the path. Hence, interchanging indices j,k,l, we have

$$H^h{}_{(j}T_{kl}{}^{i)}-H^i{}_{(j}T_{kl}{}^{h)}=0,$$

where

$$T_{jk}{}^h=H_t{}^h(\bar{\Gamma}_{jk}{}^t+\bar{\Gamma}_{kj}{}^t)-H_t{}^h(\Gamma_{jk}{}^t+\Gamma_{kj}{}^t)-(\nabla_kH_j{}^h+\nabla_jH_k{}^h).$$

By virtue of the last equation, we can take, in general

$$T_{jk}{}^h=2(u_jH_k{}^h+u_kH_j{}^h).$$

Thus the proof is completed.

In the change of (2.4), if $H_i{}^h$ is covariantly constant with respect to $\Gamma_{jk}{}^h$, then the change of (2.4) is a projective change of connections in an ordinary way. We can obtain various such changes corresponding to tensors $H_i{}^h$.

3. Pseudo projective change related to structure tensor F in an almost complex manifold

Let C^n be an n-dimensional almost complex manifold with a Riemannian metric g_{ij}, and with an almost complex structure $F_i{}^h$ satisfying the following conditions:

$$F_i{}^tF_t{}^h=-\delta_i{}^h,\quad F_j{}^lF_k{}^tg_{lt}=g_{jk},\quad F_{jk}=F_j{}^lg_{lk}=-F_{kj}. \tag{3.1}$$

From Theorem 1, next theorem is obvious.

THEOREM 2. *In an almost complex manifold with an F-connection, the pseudo projective change related to this structure tensor* $F_i{}^h$ *is a projective change of connection in an ordinary way.*

If C^n is a symmetric conformally flat space, then we can take a structure tensor $F_i{}^h$

which is a pseudo F-conformal Killing tensor defined by

$$\nabla_j F_k{}^h = q^h g_{jk} - q_k \delta_j^h + p_k F_j{}^h - p^h F_{jk},$$

where $p_i = \partial_i p$, $q_k = p_l F_k{}^l$ and p is an arbitrary scalar function [2]. Such a connection Γ, we shall call a conformally flat symmetric F-connection. In an almost complex manifold with a conformally flat symetric F-connection, if we put $H_j{}^h = F_j{}^h$, since $\bar{H}_j{}^h = -F_j{}^h$, then we have a pseudo projective change related to F. This change is given by

$$\frac{1}{2}(\bar{\Gamma}_{jk}{}^h - \bar{\Gamma}_{kj}{}^h) = \Gamma_{jk}{}^h + \left(u_j + \frac{1}{2}p_j\right)\delta_k{}^h + \left(u_k + \frac{1}{2}p_k\right)\delta_j{}^h - p^h g_{jk} + \frac{1}{2}q_j F_k{}^h + \frac{1}{2}q_k F_j{}^h. \tag{3.3}$$

Since u_j is an arbitrary covector, we can take $u_j = \frac{1}{2}p_j$ and if we put

$$\bar{\Gamma}_{jk}{}^h - \bar{\Gamma}_{kj}{}^h = -F_{jk} q^h,$$

then we have

$$\bar{\Gamma}_{jk}{}^h = \Gamma_{jk}{}^h + p_j \delta_k{}^h + p_k \delta_j{}^h - p^h g_{jk} + \frac{1}{2}F_k{}^h q_j + \frac{1}{2}F_j{}^h q_k - \frac{1}{2}F_{jk} q^h. \tag{3.4}$$

The change of (3.4) is analogous to a complex conformal connection.

By a straightforward computation, we can find the curvature tensor of $\bar{\Gamma}_{jk}{}^h$, that is,

$$\begin{aligned} \bar{R}_{jik}{}^h = R_{ijk}{}^h &- \delta_i{}^h\left(p_{jk} - \frac{1}{4}q_j q_k\right) + \delta_j{}^h\left(p_{ik} - \frac{1}{4}q_i q_k\right) \\ &- g_{ik}\left(p_i{}^h - \frac{1}{4}q_i q^h\right) + g_{ik}\left(p_j{}^h - \frac{1}{4}q_j q^h\right) \\ &+ \frac{1}{2}F_{ik} q_j{}^h - \frac{1}{2}F_{jk} q_i{}^h + \frac{1}{2}F_j{}^h q_{ik} - \frac{1}{2}F_i{}^h q_{jk} \\ &+ \frac{1}{2}F_k{}^h (q_{ij} - q_{ji}) + \frac{1}{8}(p_t p^t) F_k{}^h (F_{ji} - F_{ij}), \end{aligned} \tag{3.5}$$

where

$$p_{ij} = \nabla_i p_j - p_i p_j + \frac{1}{2} p_t p^t g_{ij}$$

$$q_{ij} = \nabla_i q_j - \frac{1}{2} p_i p_j - \frac{1}{2} q_i q_j + \frac{1}{4} p_t p^t F_{ij}.$$

Since the manifold is conformally flat, we have

$$\begin{aligned} \bar{R}_{ijk}{}^h = &\frac{1}{4}\delta_i{}^h q_j q_k - \frac{1}{4}\delta_j{}^h q_i q_k + \frac{1}{4} g_{jk} q_i q^h - \frac{1}{4} g_{jk} q_j q^h \\ &+ \frac{1}{2}F_{ik} q_j{}^h - \frac{1}{2}F_{jk} q_i{}^h + \frac{1}{2}F_j{}^h q_{ik} - \frac{1}{2}F_i{}^h q_{jk} \\ &+ \frac{1}{2}F_k{}^h (q_{ij} - q_{ji}) + \frac{1}{8}(p_t p^t) F_k{}^h (F_{ji} - F_{ij}). \end{aligned} \tag{3.6}$$

References

[1] K. Yano, *On complex conformal connections*, To appear.

[2] O. Yoon, *On conformal Killing tensors in Riemanian a manifold*, J. Korean Math. Soc., this issue, 85-87.

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