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A CURVILINEAR EXTENSION OF THE MAXIMUM PRINCIPLE FOR HARMONIC FUNCTIONS

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The object of this paper is to present a curvilinear extension of the local maximum principle for harmonic functions.

A subset on the boundary of a simply connected domain D will be called a D-conformal null set if it corresponds to a set of linear measure zero under a one-to-one conformal mapping onto the unit disc.

THEOREM: *Let D be a simply connected domain in the z-plane, which is not the whole plane, t_0 a boundary point of D, and E a D-conformal null set of B the boundary of D. If $u(z)$ is harmonic in D and bounded in the intersection of D with some neighborhood $N(t_0)$ of t_0, and at each accessible boundary point t of $B \cap N(t_0)$, possibly except those points in E, there exists an arc A_t at t on which*

$$\lim_{z\to t, z\in A_t} u(z) \leq m,$$

then

$$\lim_{z\to t, z\in D} u(z) \leq m.$$

In other words

$$\limsup_{z\to t,\ z\in D} u(z) \leq \limsup_{t\to t_0, t\in B-E} (\lim \inf_A (\limsup_{z\to t\in B, z\in A,\ A\to t} u(z))).$$

We may assume that $m=0$ by considering $u(z)-m$. By multiplying a suitable constant we may also assume that $u(z)$ is bounded by 1 in a neighborhood $N(t_0)$ of t_0.

Assume that the statement is not true. Then there exists a sequence $\{z_n\}$ in D converging to t_0, on which $u(z_n)$ converges to a positive constant, say c. Choose a positive integer N so that

$$u(z_n) > c/2 \text{ for every } n > N.$$

Let ε be a positive real number less than $c/8$. In view of [1] we can choose $r>0$ so that each accessible boundary point with its complex coordinate in $D_r \cap B$, $D_r=\{z: |z-t_0|<r\}$, there exists an arc on which $u(z) \leq \varepsilon$. Let

$$G_n^* = \{z: z\in G_n,\ u(z) > 2\varepsilon\},$$

where G_n is the component, containing z_n, of $\{z: |z-\zeta_0|<r\} \cap G$. If $\partial G_n^* \cap \partial G_n = \phi$, i.e., $G_n^* \subset G_n$ but $G_n^* \neq G_n$, for some $n>N$, then since $\limsup_{z\to t\in\partial G_n} u(z) \leq 2\varepsilon$, we obtain

$$u(z) \leq 2\varepsilon \text{ for every } z\in G_n^*$$

by the maximum principle for harmonic functions, contrary to the fact that

$$u(z) > c/2 > 4\varepsilon \text{ for } n>N,\ z\in G_n^*.$$

So we may assume that $\partial G_n \cap \partial G_n^* \neq \emptyset$ for each $n>N$.

All the possible cases will be considered.

Case 1. $G_n^* \subset G_n \cap D_r$ except finitely many $n>N$. After filling up "holes", if neces-

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sary, to make G_n^* simply connected, we map G_n^* onto a unit disc $D_w=\{w:|w|<1\}$ by a one-to-one conformal mapping f.

Let A_n (respectively A_n^*) be the set of complex coordinates of all accessible points of G_n (repectively G_n^*). Then $\partial G_n^*-A_n^*$ is a G_n^*-conformal null set. Since E is a D-conformal null set, $E\cap\partial G_n\cap\partial G_n^*$ is also a G_n^*-conformal null set. We claim that $A_n\cap A_n^*$ is a G_n^*-conformal null set. For if $t\in A_n\cap A_n^*$, there exists an arc A_a in G_n at each accessible boundary point a (with respect to G_n) with its complex coordinate t, on which $u(z)\leq\varepsilon$. And since $t\in A_n^*$, there exists an arc A_a^* in $G_n^*(\subset G_n)$ at each accessible boundary point a with its complex coordinate t, on which

$$u(z)\geq 2\varepsilon.$$

Thus, if we map G_n onto a unit disc by a one-to-one conformal mapping it corresponds to an ambiguous point of the function $u\circ f^{-1}$. But since $u\circ f^{-1}$ can have at most countably many ambiguous points by Bagemihl's ambiguous point theorem [1], $A_n\cap A_n^*$ is a conformal null set.

Thus altogether we have shown that at each point $e^{i\theta}$ of ∂D_w, possibly except those points in a set of measure zero, say E_w, there exists an arc $A_{e^{i\theta}}$ on which

$$\lim_{w\to e^{i\theta}}\sup\ (u\circ f^{-1})\ (w)\leq 2\varepsilon.$$

Hence by Noshiro's theorem [2] we obtain

$$\lim_{w\to e^{i\theta}}\sup\ (u\circ f^{-1}\ (w))\leq 2\varepsilon.$$

Since this holds for almost every point $e^{i\theta}$ on $\{w:\ w=1\}$, we obtain

$$(u\circ f^{-1})(w)\leq 2\varepsilon \text{ for } w\in D_w.$$

But for $n>N$, $f(z_n)\in D_w$,

$$(u\circ f^{-1})\ \ (f(z_n))=u(z_n)>c/2>4\varepsilon.$$

Thus we have arrived at a contradiction.

Case 2. $G_n^*\not\subset G_n\cap D_r$ for infinitely many $n>N$. Let $b_n(t)$ be the real valued function defined on

$$(\partial G_n^*\cap D_r)\cup(\partial G_n^*\cap\partial D_r)\cup(\partial D_r\cap G_n^*)$$

in the following way:

$$b_n(t)=1 \text{ for } t\in\partial G_n^*\cap\partial D_r$$
$$=0 \text{ elsewhere.}$$

Let $u_n^*(z)$ be the harmonic function obtained from Perron process with $b_n(t)$ and $G_n^*\cap D_r$. Then, by the same argument as in the case 1, we obtain

$$u(z)-u_n^*(z)\leq 2\varepsilon\quad \text{for } z\in G_n^*\cap D_r.$$

If some G_n^* contains infinitely many z_n, then ∂G_n^* contains t_0. Then since $b_n(z)$ is continuous at t_0, we have

$$\lim_{z\to t_0,\, z\in G_n^*}u_n^*(z)=b_n(t_0)=0.$$

Therefore $\lim_{n\to\infty}u(z_n)\leq\lim_{n\to\infty}u_n^*(z_n)+2\varepsilon$, i.e.,

$$\lim_{n\to\infty}u(z_n)\leq 2\varepsilon,$$

contrary to $\lim_{n\to\infty}u(z_n)=c>8\varepsilon>2\varepsilon$.

If no G_n^* contains infinitely many z_n's, we can assume that each G_n^* contains only z_n but no other z_j, $j \neq n$, by taking a suitable subsequence of $\{z_n\}$, $\{G_n^*\}$.

Let $L(r)$ be the sum of the lengths of the arcs of $G_n^* \cap \partial D_r$. Then by the well-known theorem (Carleman-Milloux Problem in [**3**]), we obtain

$$u_n^*(z_n) \leq \frac{2}{\pi} \tan^{-1} \frac{L(r)}{r \log |z_n - t_0|}.$$

As n tends to infinity, $|z_n - \zeta_0|$ tends to zero. Hence there exists a natural number N_1 such that

$$u_n^*(z_n) \leq \varepsilon \qquad \text{for every } n > N_1.$$

Let $N_2 = \max\ (N, N_1)$. Then we have

$$u(z_n) \leq u_n^*(z_n) + 2\varepsilon \leq 3\varepsilon \text{ for } n > N_2.$$

On the other hand

$$u(z_n) > c/2 > 4\varepsilon \qquad \text{for } n > N_2 > N.$$

Thus we have arrived at a contradiction again. This completes the proof of the theorem.

COROLLARY. *Let the set E of exceptional points, in the above theorem, be of logarithmic capacity zero. Then the same conclusion holds.*

Proof. Since E is of logarithmic capacity zero it is D-conformal null set (see Matsumoto [**4**]).

References

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