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REMARKS ON A MANIFOLD WITH CERTAIN (f, g, u, v, λ)-STRUCTURE

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§0. Introduction

Yano and Okumura [4] have defined a new structure called an (f, g, u, v, λ)-structure in an even-dimensional manifold as a set of a tensor field f of type $(1,1)$, a Riemannian metric g, two 1-forms u, v and a function λ satisfying certain algebraic conditions. They have showed that hypersurfaces in an almost contact metric manifold or submanifolds in an almost Hermitian manifold admit the (f, g, u, v, λ)-structure [4].

The (f, g, u, v, λ)-structure induced on hypersurface M of odd-dimensional sphere with the induced metric tensor g_{ji} and the second fundamental tensor h_{ji} satisfies

$$\nabla_j f_i{}^h = -g_{ji}u^h + \delta_j^h u_i - k_{ji}v^h + k_j{}^h v_i, \tag{0.1}$$

$$\nabla_j v_i - \nabla_i v_j = h_{it}f_j{}^t - h_{jt}f_i{}^t, \tag{0.2}$$

where $f_i{}^h$, u_i and v_i are components of f, u and v respectively, ∇_j being the operator of covariant differentiation with respect to g_{ji} ([1], [4]). (Here and in the sequel the indices $h, j, i, \cdots$ run over the range $\{1, 2, 3, \cdots, 2n\}$.)

If M is complete and has constant scalar curvature with the condition

$$h_{it}f_j{}^t - h_{jt}f_i{}^t = 0, \tag{0.3}$$

then M is a sphere or product of two spheres ([1], [2]).

Converse problems of the above theorem have been studied by Ishihara, Ki, Yano ([2], [6]) and many authors. When a complete manifold with a certain (f, g, u, v, λ)-structure has a symmetric tensor field h_{ji} of the form of (0.3), they give characterizations of even-dimensional sphere or of products of two spheres.

In the present paper we study a manifold with certain (f, g, u, v, λ)-structures which satisfy the forms of (0.1) and (0.3).

In §1, we prepare some fundamental properties of an (f, g, u, v, λ)-structure.

In §2, we find some equations for a complete manifold above to be isometric with products of two spheres.

§1. Fundamental properties on (f, g, u, v, λ)-structures

We consider a differentiable manifold M with an (f, g, u, v, λ)-structure, that is, a Riemannian manifold with metric tensor g_{ji} which admits a tensor field $f_j{}^i$ of type $(1,1)$, two 1-forms u_i and v_i, or two vectors $u^h = u_i g^{ih}$ and $v^h = v_i g^{ih}$, a differentiable function λ satisfying

$$\begin{cases} f_j{}^t f_t{}^h = -\delta_j^h + u_j u^h + v_j v^h, \\ f_j{}^t f_i{}^s g_{ts} = g_{ji} - u_j u_i - v_j v_i, \\ f_i{}^t v_t = -\lambda u_i \quad \text{or} \quad f_t{}^h v^t = \lambda u^h, \\ u_t u^t = v_t v^t = 1 - \lambda^2, \quad u_t v^t = 0, \\ f_i{}^t u_t = \lambda v_i \quad \text{or} \quad f_t{}^h u^t = -\lambda v^h, \end{cases} \tag{1.1}$$

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$f_{ji}=f_j^t g_{ti}$ being skew-symmetric. Such M is evendimension [4].

We put

$$S_{ji}^h=f_j^t\nabla_t f_i^h-f_i^t\nabla_t f_j^h-(\nabla_j f_i^t-\nabla_i f_j^t)f_t^h+U_{ji}u^h+V_{ji}v^h, \tag{1.2}$$

where $U_{ji}=\nabla_j u_i-\nabla_i u_j$, $V_{ji}=\nabla_j v_i-\nabla_i v_j$.

If the tensor S_{ji}^h vanishes, the (f,g,u,v,λ)-structure is said to be *normal* [4]. And an (f,g,u,v,λ)-structure satisfying

$$S_{ji}^h=2v_j(\nabla_i v^h-\lambda\delta_i^h)-2v_i(\nabla_j v^h-\lambda\delta_j^h) \tag{1.3}$$

is said to be *antinormal* [6].

Yano and Ki proved the following [4]

THEOREM A. *Assume that a differentiable manifold admits an antinormal* (f,g,u,v,λ)*-structure such that* $\lambda(1-\lambda^2)$ *is almost everywhere non-zero,* $\nabla_j u_i-\nabla_i u_j=2f_{ji}$. *At a point at which* $\lambda\neq 0$, *we define a tensor field* k_{ji} *of type* $(0,2)$ *by* $\nabla_j u_i+\nabla_i u_j=-2\lambda k_{ji}$. *If* u^k *and* k_{ji} *satisfy* $u^j\nabla_j u_i=0$, $\nabla_k k_{ji}-\nabla_j k_{ki}=0$, *then the manifold is isometric to* $S^n\left(\frac{1}{\sqrt{2}}\right)\times S^n\left(\frac{1}{\sqrt{2}}\right)$ *or* $\left[S^n\left(\frac{1}{\sqrt{2}}\right)\times S^n\left(\frac{1}{\sqrt{2}}\right)\right]^*$, *where* $S^n\left(\frac{1}{\sqrt{2}}\right)$ *is a sphere with radius* $\frac{1}{\sqrt{2}}$ *and* $\left[S^n\left(\frac{1}{\sqrt{2}}\right)\times S^n\left(\frac{1}{\sqrt{2}}\right)\right]^*$ *is the factor space* $S^n\left(\frac{1}{\sqrt{2}}\right)\times S^n\left(\frac{1}{\sqrt{2}}\right)/\sim$ *with Riemannian metric induced from that of* $S^n\left(\frac{1}{\sqrt{2}}\right)\times S^n\left(\frac{1}{\sqrt{2}}\right)$ *by the projection* (cf. [2]).

§2. Complete Riemannian manifolds with certain differential equations

We prove in this section

THEOREM 2.1. *Assume that a complete differentiable manifold* M *admits an* (f,g,u,v,λ)*-structure such that* $\lambda(1-\lambda^2)$ *is almost everywhere non-zero and there exists a non-zero symmetric tensor field* k_{ji} *of type* $(0,2)$ *satisfying the followings:*

$$\nabla_j f_i^h=-g_{ji}u^h+\delta_j^h u_i-k_{ji}v^h+k_j^h v_i, \tag{2.1}$$

$$k_{jt}f_i^t-k_{it}f_j^t=0, \tag{2.2}$$

$$\nabla_k k_{ji}-\nabla_j k_{ki}=0 \tag{2.3}$$

and

$$u^t\nabla_t u_j=0. \tag{2.4}$$

Then M^{2n} *is isometric to* $S^n\left(\frac{1}{\sqrt{2}}\right)\times S^n\left(\frac{1}{\sqrt{2}}\right)$ *or* $\left[S^n\left(\frac{1}{\sqrt{2}}\right)\times S^n\left(\frac{1}{\sqrt{2}}\right)\right]^*$

Proof. Transvecting (2.2) with f^{ji}, $u^j v^i$ respectively and using (1.1), we have

$$\begin{cases} k_t^t=k_{st}u^s u^t+k_{st}v^s v^t, \\ \lambda k_{st}u^s u^t+\lambda k_{st}v^s v^t=0, \end{cases} \tag{2.5}$$

and consequently

$$k_{st}u^s u^t+k_{st}v^s v^t=0 \tag{2.6}$$

because λ is non-zero almost everywhere in M.

Thus (2.5) implies that

$$k_t^t=0. \tag{2.7}$$

Transvecting (2.2) with f_k^i, and taking skew-symmetric parts of the equation obtained thus, we have

$$(k_{jt}u^t)u_k-(k_{kt}u^t)u_j+(k_{jt}v^t)v_k-(k_{kt}v^t)v_j=0,$$

from which, transvecting u^k, v^k respectively and using (2.6),

$$k_{jt}u^t=\alpha u_j+\beta v_j,\quad k_{jt}v^t=\beta u_j-\alpha v_j, \tag{2.8}$$

where α and β are defined by $(1-\lambda^2)\alpha=k_{st}u^su^t$, $(1-\lambda^2)\beta=k_{st}u^sv^t$.

Differentiating (2.2) covariantly and using (2.1) and (2.8), we have

$$0=-g_{ki}(\alpha u_j+\beta v_j)+k_{jk}u_i-k_{ki}(\beta u_j-\alpha v_j)+k_{jt}k_k^tv_i+g_{kj}(\alpha u_i+\beta v_i)$$
$$-k_{ik}u_j+k_{kj}(\beta u_i-\alpha v_i)-k_{it}k_k^tv_j,$$

from which, contracting j and k

$$2\alpha(1-n)u_i+(2n\beta-2\beta^2-2\alpha^2-2\beta+k_{st}k^{st})v_i=0.$$

Since u^k and v^k are linearly independent, we have from the equation above

$$k_{st}k^{st}=-2n\beta+2\beta+2\beta^2 \tag{2.9}$$

and $\alpha=0$. Consequently (2.8) becomes

$$k_{jt}u^t=\beta v_j,\quad k_{jt}v^t=\beta u_j. \tag{2.10}$$

On the other hand, we have from (1.1)

$$f^{ih}f_{ih}=2n-2(1-\lambda^2),$$

from which, differentiating covariantly,

$$f_{ih}(\nabla_jf^{ih})=2\lambda\nabla_j\lambda. \tag{2.11}$$

Substituting (2.1) into (2.11) and making use of (1.1) and (2.10), we obtain

$$\nabla_j\lambda=(\beta-1)v_j. \tag{2.12}$$

Differentiating both sides of the equation $f_{it}u^t=\lambda v_i$ covariantly and using (2.1) and (2.12), we find

$$(-g_{ji}u_t+g_{jt}u_i-k_{ji}v_t+k_{jt}v_i)u^t+f_i^t\nabla_ju_t=(\beta-1)v_jv_i+\lambda\nabla_jv_i,$$

from which, using (2.10)

$$\lambda\nabla_jv_i=-(1-\lambda^2)g_{ji}+(u_iu_j+v_iv_j)+f_i^t\nabla_ju_t. \tag{2.13}$$

Similarly, differentiating $f_{it}v^t=-\lambda u_i$ covariantly and using (2.1), (2.10) and (2.12), we find

$$-\lambda\nabla_ju_i=-(1-\lambda^2)k_{ji}+\beta(v_iu_j+u_iv_j)+f_i^t\nabla_jv_t. \tag{2.14}$$

From (2.13) and (2.14), we find

$$\nabla_jv_i=-k_{jt}f_i^t+\lambda g_{ji}-\frac{\lambda(\beta+1)}{1-\lambda^2}u_ju_i+\frac{1}{1-\lambda^2}u_iu^s\nabla_jv_s, \tag{2.15}$$

$$\nabla_ju_i=f_{ji}-\lambda k_{ji}+\frac{\lambda(\beta+1)}{1-\lambda^2}u_jv_i-\frac{1}{1-\lambda^2}v_iu^s\nabla_jv_s. \tag{2.16}$$

Differentiating second equation of (2.10) covariantly and substituting (2.15) and (2.16), we have

$$(\nabla_j k_i{}^t)v_t+k_i{}^t\left(-k_{js}f_t{}^s+\lambda g_{jt}-\frac{\lambda(\beta+1)}{1-\lambda^2}u_ju_t+\frac{1}{1-\lambda^2}u_tu^s\nabla_jv_s\right)$$
$$=(\nabla_j\beta)u_i+\beta(f_{ji}-\lambda k_{ji}+\frac{\lambda(\beta+1)}{1-\lambda^2}u_jv_i-\frac{1}{1-\lambda^2}v_iu^s\nabla_jv_s),$$

or, using (2.10)

$$(\nabla_j k_i{}^t)v_t-k_i{}^tk_{js}f_t{}^s-\beta f_{ji}+\lambda(\beta+1)k_{ji}$$
$$=(\nabla_j\beta)u_i+\frac{2\lambda(\beta+1)}{1-\lambda^2}u_jv_i-\frac{2\beta}{1-\lambda^2}v_iu^t\nabla_jv_t,$$

from which, taking skew-symmetric parts and using (2.2) and (2.3),

(2.17)
$$-2k_i{}^tk_{ts}f_j{}^s-2\beta f_{ji}$$
$$=(\nabla_j\beta)u_i-(\nabla_i\beta)u_j+\frac{2\lambda\beta(\beta+1)}{1-\lambda^2}(u_jv_i-u_iv_j)-\frac{2\beta}{1-\lambda^2}(v_iu^t\nabla_jv_t-v_ju^t\nabla_iv_t).$$

Transvecting (2.17) with u^i, v^i respectively and using (2.4) and (2.10), we have

(2.18)
$$(1-\lambda^2)\nabla_j\beta=(u^t\nabla_t\beta)u_j,$$

(2.19)
$$2\beta u^t\nabla_jv_t=-(v^t\nabla_t\beta)u_j+\frac{2\beta}{1-\lambda^2}(u^tv^s\nabla_sv_t)v_j.$$

Substituting (2.18) and (2.19) into (2.17), we obtain

(2.20)
$$-2k_i{}^tk_{ts}f_j{}^s-2\beta f_{ji}$$
$$=\frac{1}{1-\lambda^2}\{2\beta\lambda(\beta+1)+v^t\nabla_t\beta\}(v_iu_j-v_ju_i),$$

from which, transvecting u^iv^j

(2.21)
$$v^t\nabla_t\beta=0.$$

Thus (2.20) becomes

$$(1-\lambda^2)\{k_i{}^tk_{ts}f_j{}^s+\beta f_{ji}\}=\lambda\beta(\beta+1)(v_ju_i-v_iu_j).$$

Transvecting the last equation with $f_k{}^j$ and using (1.1) and (2.10), we get

$$(1-\lambda^2)\{k_i{}^tk_{ts}(-\delta_k^s+u_ku^s+v_kv^s)+\beta(-g_{ik}+u_iu_k+v_iv_k)\}$$
$$=-\lambda^2\beta(\beta+1)(u_iu_k+v_iv_k),$$

that is,

(2.22)
$$(1-\lambda^2)(k_j{}^tk_{ti}+\beta g_{ji})=\beta(\beta+1)(u_iu_k+v_iv_k).$$

Since k_{ji} is non-zero tensor, by (2.9) we can easily see that β is not 0 or 1 on M. Using (2.21), equations (2.18) and (2.19) can be respectively written as the forms

(2.23)
$$\nabla_j\beta=Au_j,$$

(2.24)
$$u^t\nabla_jv_t=Bv_j$$

for certain differentiable functions A and B on M.

Differentiating (2.23) covariantly and using (2.16) and (2.24), we find

$$\nabla_j\nabla_i\beta=(\nabla_jA)u_i+A(f_{ji}-\lambda k_{ji}+\frac{\lambda(\beta+1)}{1-\lambda^2}u_jv_i-\frac{B}{1-\lambda^2}v_jv_i),$$

from which

(2.25) $$0=(\nabla_j A)u_i-(\nabla_i A)u_j+A\{2f_{ji}+\frac{\lambda(\beta+1)}{1-\lambda^2}(u_j v_i-u_i v_j)\}.$$

Transvecting (2.25) with $u^i v^j$, f^{ji} respectively, we have

$$v^t\nabla_t A+A\lambda(1-\beta)=0,$$
$$\lambda v^t\nabla_t A+A\{2n-2(1-\lambda^2)-\lambda^2(\beta+1)\}=0$$

because $1-\lambda^2$ is non-zero almost everywhere on M.

From the last two equations, we find $A=0$ and consequently $\beta=$constant by virtue of $n>1$.

Differentiating (2.12) covariantly and using $\beta=$constant, we find $\nabla_k\nabla_j\lambda=(\beta-1)\nabla_k v_j$, from which, using (2.15) and (2.24),

$$(\beta-1)B(u_k v_j-u_j v_k)=0.$$

Since β can not be 1, we have $B=0$ and consequently

(2.26) $$u^t\nabla_j v_t=0.$$

Differentiating (2.22) covariantly and using $\beta=$constant and (2.12), we find

$$-2\lambda(\beta-1)v_k(k_{jt}k_i^t+\beta g_{ji})+(1-\lambda^2)\{(\nabla_k k_{jt})k_i^t+k_j^t(\nabla_k k_{it})\}$$
$$=\beta(\beta+1)\{(\nabla_k u_j)u_i+u_j(\nabla_k u_i)+(\nabla_k v_j)v_i+v_j(\nabla_k v_i)\},$$

from which, transvecting g^{kj} and using (2.3) and (2.4),

$$-2\lambda(\beta-1)\beta(\beta+1)v_i+(1-\lambda^2)\left\{(\nabla_t k_s^t)k_i^s-\frac{1}{2}\nabla_i(k_{jt}k^{jt})\right\}$$
$$=\beta(\beta+1)\{(\nabla_t u^t)u_i+(\nabla_t v^t)v_i+v^t\nabla_t v_i\},$$

or, using $\beta=$constant, (2.7) and (2.9),

(2.27) $$-2\lambda(\beta-1)\beta(\beta+1)v_i=\beta(\beta+1)\{(\nabla_t u^t)u_i+(\nabla_t v^t)v_i+v^t\nabla_t v_i\}.$$

On the other hand, by (2.7), (2.10), (2.15), (2.16) and (2.26) we can easily verify that $\nabla_t u^t=0$, $\nabla_t v^t=2n\lambda-\lambda(\beta+1)$ and $v^t\nabla_t v_i=\lambda(1-\beta)v_i$.

Substituting these equations into (2.27), we see that $\beta(\beta+1)=0$. Since β can not be 0, we have $\beta=-1$ on M. Therefore, (2.15), (2.16) and (2.26) imply that

(2.28) $$\nabla_j u_i=f_{ji}-\lambda k_{ji},\quad \nabla_j v_i=-k_{jt}f_i^t+\lambda g_{ji},$$

from which,

$$\nabla_j u_i-\nabla_i u_j=2f_{ji},\quad \nabla_j u_i+\nabla_i u_j=-2\lambda k_{ji}.$$

Substituting (2.1) and (2.28) into (1.2) and using (2.2), we find (1.3). Thus the assumptions of Theorem A are all satisfied and consequently the conclusions of the theorem are valid.

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