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SUBMANIFOLDS OF CODIMENSION 2 WITH A COMPLEMENTED f-STRUCTURE

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§0. Introduction

A structure called an (f, g, u, v, λ)-structure induced on a submanifold of codimension 2 of an Hermitian manifold has been studied by many authors (cf. [1], [9]). The submanifolds of codimension 2 in an even-dimensional Euclidean space in terms of this structure have been studied by the present authors [2], [3], [4], [8], Okumura [5], [10], Pak [3], [6], Yano [8], [10] and the others.

In the present paper, we study submanifolds of codimension 2 with a complemented f-structure in an even-dimensional Euclidean space under the assumptions such that h and f commute and k and f anti-commute, where h and k are the second fundamental tensors of the submanifold.

In §1, we consider submanifolds of codimension 2 of an even-dimensional Euclidean space regarded as a flat Kaehlerian manifold and find several equations which the complemented f-structure satisfies.

In §2, we prepare several lemmas on the submanifold under the assumptions stated above.

In §3 and §4, we prove the submanifolds of codimension 2 with a complemented f-structure in an even-dimensional Euclidean space are locally symmetric under the additional conditions.

§1. Preliminaries

We denote by X the position vector starting from the origin and ending at a point P in a Euclidean space E of dimension $2n+2$. The E being even-dimensional, it can be regarded as a flat Kaehlerian manifold with the numerical structure tensor F: $F^2=-I$, where I denotes the unit tensor and $FY\cdot FZ=Y\cdot Z$ for arbitrary vector fields Y and Z, where the dot denotes the inner product of vectors of E.

We consider a submanifold M of codimension 2 in E and assume that M is covered by coordinate neighborhoods $\{U: x^h\}$, where here and in the sequel the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, 2n\}$. If we put $X_i=\partial_i X$, $\partial_i=\partial/\partial x^i$, then we see that X_i are $2n$ linearly independent local vector fields tangent to M and defined in U. The Riemannian metric g induced on M has local components of the form $g_{ji}=X_j\cdot X_i$ in U. We assume that we can take along M two globally defined mutually orthogonal unit normals C and D to M in such a way that X_i, C and D give the positive orientation of E.

Throughout the present paper, we assume that the transforms FX_i of X_i by the complex structure F are linear combinations of X_h, C and D, that is,

$$FX_i=f_i^{\ h}X_h+u_iC+v_iD, \tag{1.1}$$

where $f_i^{\ h}$ are components of a tensor field f of type $(1,1)$, u_i and v_i are components

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of two 1-forms, all globally defined on M and the transforms FC and FD of C and D by F can be expressed as

$$FC=-u^hX_h, \tag{1.2}$$

$$FD=-v^hX_h \tag{1.3}$$

respectively, where $u^h=u_ig^{ih}$, $v^h=v_ig^{ih}$.

If we apply F to (1.1), (1.2) and (1.3) respectively and using $F^2=-I$, (1.1), (1.2) and (1.3), we have

$$\begin{aligned} &f_i^t f_t^h=-\delta_i^h+u_iu^h+v_iv^h,\\ &u_tf_i^t=v_tf_i^t=0, \qquad f_t^hu^t=f_t^hv^t=0,\\ &u_tu^t=v_tv^t=1, \qquad u_tv^t=0. \end{aligned} \tag{1.4}$$

We also have, from (1.1),

$$g_{ts}f_j^tf_i^s=g_{ji}-u_ju_i-v_jv_i \tag{1.5}$$

by virtue of $FX_j\cdot FX_i=X_j\cdot X_i=g_{ji}$. From (1.4), we can easily see that

$$f^3+f=0. \tag{1.6}$$

The structure induced on M by such a set of a tensor field f of type (1,1), a Riemannian metric g and two 1-forms u and v satisfying (1.4) and (1.5) is a complemented f-structure (cf. [1] etc.). In the sequel, we consider only submanifold M with a complemented f-structure.

We denote by $\{{}_j{}^h{}_i\}$ the Christoffel symbols formed with g_{ji} and by ∇_j the operator of covariant differentiation with respect to $\{{}_j{}^h{}_i\}$. Then, for the submanifold M, the equations of Gauss are

$$\nabla_jX_i=h_{ji}C+k_{ji}D, \tag{1.7}$$

h_{ji} and k_{ji} being the components of the second fundamental tensors with respect to the normals C and D respectively, where $\nabla_jX_i=\partial_jX_i-\{{}_j{}^h{}_i\}X_h$. The equations of Weingarten are

$$\begin{aligned} &\nabla_jC=-h_j^hX_h+l_jD,\\ &\nabla_jD=-k_j^hX_h-l_jC, \end{aligned} \tag{1.8}$$

h_j^h and k_j^h being defined respectively by $h_j^h=h_{ji}g^{ih}$, $k_j^h=k_{ji}g^{ih}$, which are respectively the components of tensor fields h and k of type (1,1) and l_j components of the third fundamental tensor, where $\nabla_jC=\partial_jC$, $\nabla_jD=\partial_jD$.

Now, differentiating (1.1) covariantly and taking account of $\nabla_jF=0$ and of equations of Gauss and Weingarten, we find

$$\nabla_jf_i^h=-h_{ji}u^h+h_j^hu_i-k_{ji}v^h+k_j^hv_i, \tag{1.9}$$

$$\nabla_ju_i=-h_{jt}f_i^t+l_jv_i, \tag{1.10}$$

$$\nabla_jv_i=-k_{jt}f_i^t-l_ju_i. \tag{1.11}$$

Similarly we have, from (1.2),

$$k_{ji}u^i=h_{ji}v^i. \tag{1.12}$$

In the sequel, we need the structure equations of the submanifold M, that is, the following equations of Gauss

(1.13) $$K_{kjih}=h_{kh}h_{ji}-h_{jh}h_{ki}+k_{kh}k_{ji}-k_{jh}k_{ki},$$

where K_{kjih} are covariant components of the curvature tensor of M, and equations of Codazzi and Ricci

(1.14) $$\nabla_k h_{ji}-\nabla_j h_{ki}-l_k k_{ji}+l_j k_{ki}=0,$$

(1.15) $$\nabla_k k_{ji}-\nabla_j k_{ki}+l_k h_{ji}-l_j h_{ki}=0,$$

(1.16) $$\nabla_j l_i-\nabla_i l_j+h_{jt}k_i^t-h_{it}k_j^t=0.$$

§2. Lemmas on the submanifold M

We suppose that h_j^h commute with f_j^h, i.e.,

(2.1) $$f_j^t h_t^h-h_j^t f_t^h=0,$$

which is equivalent to

(2.2) $$h_{jt}f_i^t+h_{it}f_j^t=0,$$

that is, $h_{jt}f_i^t$ is skew-symmetric. We suppose also that k_j^h anti-commute with f_j^h, i.e.,

(2.3) $$f_j^t k_t^h+k_j^t f_t^h=0,$$

which is equivalent to

(2.4) $$k_{jt}f_i^t-k_{it}f_j^t=0,$$

that is, $k_{jt}f_i^t$ is symmetric.

We first prove

LEMMA 2.1. *Let M be a submanifold of codimension 2 with a complemented f-structure in an even-dimensional Euclidean space such that (2.1) and (2.3) are satisfied. Then we have*

(2.5) $$h_{jt}u^t=pu_j+\alpha v_j,$$

(2.6) $$h_{jt}v^t=\alpha u_j+\beta v_j=k_{jt}u^t,$$

(2.7) $$k_{jt}v^t=\beta u_j+\gamma v_j,\quad k_t^t=\alpha+\gamma,$$

p, α, β and γ being given respectively by

$$p=h_{ts}u^t u^s,\quad \alpha=h_{ts}u^t v^s=k_{ts}u^t u^s,$$

$$\beta=h_{ts}v^t v^s=k_{ts}u^t v^s,\quad \gamma=k_{ts}v^t v^s.$$

Proof. Transvecting (2.2) with $u^j f_k^i$, we get (2.5). Transvecting (2.2) with $v^j f_k^i$ and using (1.12), we have (2.6). Transvecting (2.4) with $v^j f_k^i$, we find the first equation of (2.7) and transvecting (2.4) with f^{ji}, we get

$$2f^{ji}k_{jt}f_i^t=0,$$

from which, $k_t^t=\alpha+\gamma$. Consequently Lemma 2.1 is proved.

LEMMA 2.2. *Under the same assumptions as those stated in Lemma 2.1, we have*

(2.8) $$k_l^t h_{kt}-h_l^t k_{kt}=(2\alpha^2+\beta^2-\beta p-\alpha k_t^t)(u_l v_k-v_l u_k),$$

(2.9) $$\alpha h_{ji}=\alpha p u_j u_i+\alpha^2(u_j v_i+v_j u_i)+\alpha\beta v_j v_i,$$

(2.10) $$\alpha h_t^t=\alpha(p+\beta).$$

Proof. Differentiating (2.6) covariantly, we find

$$(\nabla_k h_{jt})v^t + h_j^t(\nabla_k v_t) = (\nabla_k \alpha)u_j + \alpha(\nabla_k u_j) + (\nabla_k \beta)v_j + \beta(\nabla_k v_j),$$

from which, taking skew-symmetric parts and using (1.10), (1.11), (1.14), (2.4), (2.5) and (2.7),

$$\begin{aligned}&\{(\beta-p)u_j + (k_t^t - 2\alpha)v_j\}l_k - \{(\beta-p)u_k + (k_t^t - 2\alpha)v_k\}l_j - h_j^t k_{ks} f_t^s + h_k^t k_{js} f_t^s \\ &= (\nabla_k \alpha)u_j - (\nabla_j \alpha)u_k + (\nabla_k \beta)v_j - (\nabla_j \beta)v_k \\ &\quad -2\alpha h_{kt} f_j^t + \alpha(l_k v_j - l_j v_k) + \beta(l_j u_k - l_k u_j).\end{aligned} \tag{2.11}$$

Transvecting (2.11) with u^k and using (2.5) and (2.6), we find

$$\begin{aligned}\nabla_j \alpha = &\{(p-2\beta)l_t u^t + u^t \nabla_t \alpha\}u_j \\ &+ \{(3\alpha - k_s^s)l_t u^t + u^t \nabla_t \beta\}v_j \\ &+ (2\beta - p)l_j.\end{aligned} \tag{2.12}$$

Transvecting also (2.11) with v^k and taking account of (2.6) and (2.7), we have

$$\begin{aligned}\nabla_j \beta = &\{(p-2\beta)l_t v^t + v^t \nabla_t \alpha\}u_j \\ &+ \{(3\alpha - k_s^s)l_t v^t + v^t \nabla_t \beta\}v_j \\ &+ (k_t^t - 3\alpha)l_j.\end{aligned} \tag{2.13}$$

Substituting (2.12) and (2.13) into (2.11), we get

$$\begin{aligned}&-h_j^t k_{ks} f_t^s + h_k^t k_{js} f_t^s + 2\alpha h_{kt} f_j^t \\ &= \{[(3\alpha - k_s^s)l_t u^t + u^t \nabla_t \beta] - [(p-2\beta)l_t v^t \\ &\quad + v^t \nabla_t \alpha]\}(u_j v_k - v_j u_k).\end{aligned} \tag{2.14}$$

Transvecting (2.14) with u^j and using (2.5) and (2.6), we find

$$(3\alpha - k_s^s)l_t u^t + u^t \nabla_t \beta = (p-2\beta)l_t v^t + v^t \nabla_t \alpha. \tag{2.15}$$

Substituting (2.15) into (2.14), we find

$$h_j^t k_{ks} f_t^s - h_k^t k_{js} f_t^s = 2\alpha h_{kt} f_j^t, \tag{2.16}$$

which is equivalent to

$$h_s^t k_k^s f_{jt} - h_k^t k_{ts} f_j^s = 2\alpha h_{kt} f_j^t. \tag{2.17}$$

Transvecting (2.17) with f_i^j and using (2.5)$\sim$(2.7), we get

$$\begin{aligned}&k_i^t h_{kt} - h_i^t k_{kt} + 2\alpha h_{ik} \\ &= 2\alpha p u_i u_k - (p\beta + \alpha k_t^t - \beta^2 - 4\alpha^2)u_i v_k \\ &\quad - (\beta^2 - \beta p - \alpha k_t^t)v_i u_k + 2\alpha\beta\, v_i v_k,\end{aligned} \tag{2.18}$$

from which, taking skew-symmetric parts and symmetric parts respectively, it follows that

$$\begin{aligned}&2(k_i^t h_{kt} - h_i^t k_{kt}) \\ &= 2(2\alpha^2 + \beta^2 - p\beta - \alpha k_t^t)(u_i v_k - v_i u_k).\end{aligned}$$

and

$$4\alpha h_{ik} = 4\alpha p u_i u_k + 4\alpha^2(u_i v_k + v_i u_k) + 4\alpha\beta v_i v_k.$$

Thus, we get (2.8) and (2.9). Contracting (2.9) with i and j, we have (2.10).

Therefore, Lemma 2.2 is proved.

LEMMA 2.3. *Under the same assumptions as those stated in Lemma* 2.1, *we have*

$$h_{jt}h_i^t-ph_{ji}=\alpha^2u_ju_i+\alpha\beta(u_jv_i+v_ju_i)+(\alpha^2+\beta^2-\beta p)v_jv_i \tag{2.19}$$

$$\begin{aligned}k_{jt}k_i^t+\beta h_{ji}=&(\alpha^2+\beta^2+\beta p)u_ju_i\\&+(\alpha\beta+\beta k_t^t)(u_jv_i+v_ju_i)\\&+\{2\beta^2+(k_t^t-\alpha)^2\}v_jv_i.\end{aligned} \tag{2.20}$$

Proof. Differentiating (2.5) covariantly, we find

$$\begin{aligned}&(\nabla_kh_{jt})u^t+h_j^t(\nabla_ku_t)\\&=(\nabla_kp)u_j+p(\nabla_ku_j)+(\nabla_k\alpha)v_j+\alpha(\nabla_kv_j),\end{aligned}$$

from which, taking skew-symmetric parts and using (1.14), (2.5), (2.6) and (2.7), we find

$$\begin{aligned}&2(\alpha u_j+\beta v_j)l_k-2(\alpha u_k+\beta v_k)l_j+2h_k^th_{js}f_t^s\\&=(\nabla_kp)u_j-(\nabla_jp)u_k+(\nabla_k\alpha)v_j-(\nabla_j\alpha)v_k\\&\quad-2ph_{kt}f_j^t+p(l_kv_j-l_jv_k)+\alpha(l_ju_k-l_ku_j).\end{aligned} \tag{2.21}$$

Transvecting (2.21) with u^k, we find

$$\begin{aligned}\nabla_jp=&(u^t\nabla_tp-3\alpha l_tu^t)u_j\\&+\{u^t\nabla_t\alpha+(p-2\beta)l_tu^t\}v_j+3\alpha l_j.\end{aligned} \tag{2.22}$$

Transvecting also (2.21) with v^k, we find

$$\begin{aligned}\nabla_j\alpha=&(v^t\nabla_tp-3\alpha l_tv^t)u_j\\&+\{v^t\nabla_t\alpha+(p-2\beta)l_tv^t\}v_j+(2\beta-p)l_j.\end{aligned} \tag{2.23}$$

Substituting (2.22) and (2.23) into (2.21), we have

$$\begin{aligned}&2h_k^th_{js}f_t^s+2ph_{kt}f_j^t\\&=\{[u^t\nabla_t\alpha+(p-2\beta)l_tu^t]-[v^t\nabla_tp-3\alpha l_tv^t]\}(u_jv_k-v_ju_k),\end{aligned}$$

from which, transvecting with u^j,

$$u^t\nabla_t\alpha+(p-2\beta)l_tu^t=v^t\nabla_tp-3\alpha l_tv^t,$$

and substituting this into the last equation,

$$h_k^th_{js}f_t^s=-ph_{kt}f_j^t, \tag{2.24}$$

or, equivalently

$$h_k^th_{ts}f_j^s=ph_{kt}f_j^t. \tag{2.25}$$

Transvecting (2.25) with f_i^j and using (2.5) and (2.6), we have (2.19). From (2.7), equation (2.20) will be proved in a similar way.

§ 3. Submanifolds whose normal bundle is trivial.

In this section, we consider the submanifold M whose normal bundle is trivial. We first prove

LEMMA 3.1. *Let M be a submanifold of codimension 2 with a complemented f-structure in an even-dimensional Euclidean space such that the normal bundle is trivial, and*

(2.1) and (2.3) are satisfied. Then we have

$$2\alpha^2+\beta^2=\beta p+\alpha k_t{}^t, \tag{3.1}$$

$$2\beta p h_{ji}=2p^2\beta u_j u_i+2\alpha\beta p(u_j v_i+v_j u_i)+2p\beta^2 v_j v_i+\beta l_t v^r h_{is} f_j{}^s+l_t u^t h_{es} k_j{}^e f_i{}^s, \tag{3.2}$$

$$\beta p\{h_t{}^t-(p+\beta)\}=0. \tag{3.3}$$

Proof. By the assumption and (1.16), we have

$$h_{jt}k_i{}^t=h_{it}k_j{}^t. \tag{3.4}$$

From this and (2.8), we get

$$(2\alpha^2+\beta^2-\beta p-\alpha k_t{}^t)(u_i v_k-v_i u_k)=0,$$

which implies (3.1). Next, differentiating (2.2) covariantly, we find

$$(\nabla_k h_{jt})f_i{}^t+(\nabla_k h_{it})f_j{}^t=-h_{jt}(\nabla_k f_i{}^t)-h_{it}(\nabla_k f_j{}^t), \tag{3.5}$$

from which, taking skew-symmetric parts and using (1.14),

$$\begin{aligned}(l_k k_{jt}-l_j k_{kt})f_i{}^t+(\nabla_k h_{it})f_j{}^t-(\nabla_j h_{it})f_k{}^t\\=-h_{jt}(\nabla_k f_i{}^t)+h_{kt}(\nabla_j f_i{}^t)-h_{it}(\nabla_k f_j{}^t-\nabla_j f_k{}^t),\end{aligned} \tag{3.6}$$

from which, interchanging the indices i and j,

$$\begin{aligned}(\nabla_k h_{jt})f_i{}^t-(\nabla_i h_{jt})f_k{}^t+(l_k k_{it}-l_i k_{kt})f_j{}^t\\=-h_{it}(\nabla_k f_j{}^t)+h_{kt}(\nabla_i f_j{}^t)-h_{jt}(\nabla_k f_i{}^t-\nabla_i f_k{}^t).\end{aligned} \tag{3.7}$$

Computing (3.5) − (3.6) − (3.7) and using (1.9) and (1.14), we obtain

$$\begin{aligned}&2l_j k_{kt} f_i{}^t-2l_k k_{jt} f_i{}^t+2(\nabla_i h_{jt})f_k{}^t\\&=2h_{kt}u^t h_{ji}+2h_{kt}v^t k_{ji}-2h_{it}h_j{}^t u_k\\&\quad+(h_{it}k_k{}^t-h_{kt}k_i{}^t)v_j+(h_{jt}k_k{}^t-h_{kt}k_j{}^t)v_i\\&\quad-(h_{it}k_j{}^t+h_{jt}k_i{}^t)v_k,\end{aligned}$$

from which, transvecting with v^k and using (2.5), (2.6) and (3,4),

$$h_i{}^t k_{jt}-l_s v^s k_{jt} f_i{}^t=\alpha h_{ji}+\beta k_{ji}. \tag{3.8}$$

Differentiating (2.4) covariantly, we find

$$(\nabla_k k_{jt})f_i{}^t+k_{jt}(\nabla_k f_i{}^t)=(\nabla_k k_{it})f_j{}^t+k_{it}(\nabla_k f_j{}^t), \tag{3.9}$$

from which, taking skew-symmetric parts and using (1.15),

$$\begin{aligned}(l_j h_{kt}-l_k h_{jt})f_i{}^t+k_{jt}(\nabla_k f_i{}^t)-k_{kt}(\nabla_j f_i{}^t)\\=(\nabla_k k_{it})f_j{}^t-(\nabla_j k_{it})f_k{}^t+k_{it}(\nabla_k f_j{}^t-\nabla_j f_k{}^t).\end{aligned} \tag{3.10}$$

Subtracting (3.10) to (3.9) and using (1.9), we obtain

$$\begin{aligned}&(\nabla_k k_{jt})f_i{}^t+(l_k h_{jt}-l_j h_{kt})f_i{}^t-(\nabla_j k_{it})f_k{}^t\\&=(k_{kt}h_{ji}-k_{it}h_{jk})u^t+k_{it}h_j{}^t u_k-k_{kt}h_j{}^t u_i\\&\quad+(k_{kt}k_{ji}-k_{it}k_{jk})v^t+k_{it}k_j{}^t v_k-k_{kt}k_j{}^t v_i,\end{aligned}$$

from which, transvecting with u^i and using (2.5) and (2.6),

$$\begin{aligned}&k_{kt}h_j{}^t+\alpha h_{jk}+\beta k_{jk}\\&=u^e(\nabla_j k_{et})f_k{}^t+2(\alpha p+\alpha\beta)u_j u_k\\&\quad+2(\alpha^2+\beta^2)(u_j v_k+v_j u_k)+2\beta k_t{}^t v_j v_k,\end{aligned} \tag{3.11}$$

from which, taking skew-symmetric parts,

$$u^e(\nabla_j k_{et}) f_k{}^t = u^e(\nabla_k k_{et}) f_j{}^t. \tag{3.12}$$

Substituting (3.8) into (3.11), we get

$$\begin{aligned} 2k_{kt}h_j{}^t = & l_s v^s k_{kt} f_j{}^t + u^e(\nabla_k k_{et}) f_j{}^t \\ & + 2(\alpha p + \alpha\beta) u_j u_k + 2(\alpha^2 + \beta^2)(u_j v_k + v_j u_k) \\ & + 2\beta k_t{}^t v_j v_k. \end{aligned} \tag{3.13}$$

Transvecting (2.20) with $2h_k{}^i$, we get

$$\begin{aligned} 2k_j{}^t k_{ti} h_k{}^i + 2\beta h_{ji} h_k{}^i = & 2(p\alpha^2 + p\beta^2 + \beta p^2 + \alpha^2\beta + \alpha\beta k_t{}^t) u_j u_k \\ & + 2(\alpha^3 + \alpha\beta^2 + \alpha\beta p + \alpha\beta^2 + \beta^2 k_t{}^t) u_j v_k \\ & + 2\{\alpha\beta p + \beta p k_t{}^t + 2\alpha\beta^2 + \alpha(k_t{}^t)^2 \\ & - 2\alpha^2 k_t{}^t + \alpha^3\} v_j u_k \\ & + 2\{\alpha^2\beta + \alpha\beta k_t{}^t + 2\beta^3 + \beta(k_t{}^t)^2 \\ & - 2\alpha\beta k_t{}^t + \alpha^2\beta\} v_j v_k. \end{aligned} \tag{3.14}$$

Substituting (2.19) and (3.13) into (3.14), we obtain

$$\begin{aligned} & 2\beta p h_{jk} + l_e v^e k_j{}^t k_{ts} f_k{}^s + u^e(\nabla_k k_{es}) k_j{}^t f_t{}^s \\ & = 2\{\beta p^2 u_j u_k + \alpha\beta p(u_j v_k + v_j u_k) + \beta^2 p v_j v_k\} \end{aligned} \tag{3.15}$$

Substituting (1.15) and (2.20) into (3.15), using (2.2) and (3.12), we have (3.2). Contracting (3.2) with i and j, we have (3.3). This completes the proof of Lemma 3.1.

LEMMA 3.2. *Under the same assumptions as those stated in Lemma 3.1, we have*

$$\alpha K = 0 \tag{3.16}$$

and

$$\beta K = 0, \tag{3.17}$$

where $K = g^{ji}K_{ji}$ is scalar curvature of M.

Proof. From (1.13), we get

$$K_{ji} = g^{kh}K_{kjih} = h_t{}^t h_{ji} - h_{jt}h_i{}^t + k_t{}^t k_{ji} - k_{jt}k_i{}^t, \tag{3.18}$$

from which, transvecting with g^{ji},

$$K = g^{ji}K_{ji} = (h_t{}^t)^2 - h_{st}h^{st} + (k_t{}^t)^2 - k_{st}k^{st}. \tag{3.19}$$

Substituting (2.19) and (2.20) into (3.19) and using (3.1), we get

$$K = \{h_t{}^t + 2\beta\}\{h_t{}^t - (\beta + p)\}. \tag{3.20}$$

From (2.10) and (3.20), we have (3.16). From (3.6) and (3.20), we find

$$\beta p K = 0. \tag{3.21}$$

Now, we assume that the set $N = \{x \in M;\ (\beta K)_x \neq 0\}$ is not void, then $p = 0$ on N.

By the way, from (2.5) and (2.6), we find

$$\begin{aligned} & \{h_{ji} - p u_j u_i - \alpha(u_j v_i + v_j u_i) - \beta v_j v_i\} \cdot \{h^{ji} \\ & - p u^j u^i - \alpha(u^j v^i + v^j u^i) - \beta v^j v^i\} = p\{h_t{}^t - (\beta + p)\}, \end{aligned} \tag{3.22}$$

from which,

$$h_{ji}=\alpha(u_jv_i+v_ju_i)+\beta v_jv_i \text{ on } N.$$

Transvecting the last equation with g^{ji}, we find

$$h_t^t=\beta \text{ on } N. \tag{3.23}$$

Substituting (3.23) into (3.20), we obtain

$$K=0,$$

and consequently $\beta K=0$ on N. This contradicts to the defini tion of the set N. Hence, $\beta K=0$ on the whole space M. This completes the proof of Lemma 3.2.

We prove the following

THEOREM 3.3. *Let M be a submanifold of codimension 2 with a complemented f-structure in an even-dimensional Euclidean space such that the normal bundle is trivial, and* (2.1) *and* (2.3) *are satisfied. If the scalar curvature of M is constant, then M is locally symmetric.*

Proof. Since K=constant, we may consider two cases $K\neq 0$ or $K=0$. First, in the case of $K\neq 0$, from (3.16) and (3.17), we have $\alpha=0$ and $\beta=0$.

By the way, from (2.6) and (2.7), we have

$$\begin{aligned}&\{k_{ji}-\alpha u_ju_i-\beta(u_jv_i+v_ju_i)-(k_t^t-\alpha)v_jv_i\}\ \{k^{ji}\\ &-\alpha u^ju^i-\beta(u^jv^i+v^ju^i)-(k_t^t-\alpha)v^jv^i\}\\ &=-\beta\{h_t^t-(\beta+p)\},\end{aligned} \tag{3.24}$$

from which

$$k_{ji}=k_t^tv_jv_i,\quad k_{jt}h_k^t=0. \tag{3.25}$$

From (2.19), we have,

$$h_{jt}h_i^t=ph_{ji}, \tag{3.26}$$

by virtue of $\alpha=0$ and $\beta=0$. Thus h_j^h has at most two eigenvalues 0 and p.

Substituting (3.25) into (1.13), we find

$$K_{kjih}=h_{ji}h_{kh}-h_{ki}h_{jh}. \tag{3.27}$$

If we put $h_t^t=mp$, m being the multiplicity of the eigenvalue p, then we have

$$K=h_t^t(h_t^t-p)=m(m-1)p^2. \tag{3.28}$$

Since $p\neq 0$, h_j^h has exactly two eigenvalues, thus m=const. ($m>1$) and consequently p=constant (cf. [7]).

Differentiating (3.26) covariantly, we find

$$(\nabla_kh_{jt})h_i^t+h_j^t(\nabla_kh_{it})=p(\nabla_kh_{ji}), \tag{3.29}$$

from which, taking skew-symmetric parts with respect to j and k and using (1.14) and second equation of (3.25),

$$h_j^t(\nabla_kh_{it})-h_k^t(\nabla_jh_{it})=p(l_kk_{ji}-l_jk_{ki}), \tag{3.30}$$

from which, interchanging the indices i and j,

$$h_i^t(\nabla_kh_{jt})-h_k^t(\nabla_ih_{jt})=p(l_kk_{ij}-l_ik_{kj}). \tag{3.31}$$

Computing (3.29)−(3.30)+(3.31) and using (1.14) and second equation of (3.25), we obtain

$$2(\nabla_k h_{jt})h_i^t = p(\nabla_k h_{ji} + l_j k_{ki} - l_i k_{jk}),$$

from which, transvecting with u^i,

$$2p(\nabla_k h_{jt})u^t = p(\nabla_k h_{jt})u^t - p l_t u^t k_{jk},$$

that is,

$$(\nabla_k h_{jt})u^t = -(l_t u^t)k_{jk}. \tag{3.32}$$

Differentiating $h_{jt}u^t = pu_j$ covariantly, we get

$$(\nabla_k h_{jt})u^t + h_j^t(\nabla_k u_t) = p(\nabla_k u_j). \tag{3.33}$$

Substituting (3.32) into (3.33) and using (1.10), we get

$$-(l_t u^t)k_{jk} - h_j^t h_{ks} f_t^s + l_k h_j^t v_t$$
$$= -p h_{kt} f_j^t + p l_k v_j,$$

from which, transvecting with v^j,

$$-(l_t u^t)k_s^s v_k = p l_k, \tag{3.34}$$

which implies

$$l_t u^t = 0, \tag{3.35}$$

from which, using (3.34) and $p \neq 0$,

$$l_k = 0. \tag{3.36}$$

Taking account of (3.36), (3.30) becomes

$$h_j^t(\nabla_k h_{it}) - h_k^t(\nabla_j h_{it}) = 0,$$

from which, interchanging the indices i and k,

$$h_j^t(\nabla_i h_{kt}) - h_i^t(\nabla_j h_{kt}) = 0, \tag{3.37}$$

Adding (3.29) and (3.37), we find

$$2h_j^t(\nabla_k h_{it}) = p(\nabla_k h_{ji}). \tag{3.38}$$

Transvecting (3.38) with h_i^j and using (3.26), we get

$$h_t^t(\nabla_k h_{it}) = 0.$$

Thus, (3.38) implies that

$$\nabla_k h_{ji} = 0. \tag{3.39}$$

Hence, differentiating (3.27) covariantly and using (3.39), we have

$$\nabla_l K_{kjih} = 0, \tag{3.40}$$

consequently, M is locally symmetric.

Next, in the case of $K=0$, we assume that the set $N = \{x \in M : (h_t^t - (\beta + p))_x \neq 0\}$ is not void. Then, from (2.10), (3.1) and (3.4), we have

$$\alpha = 0 \text{ and } \beta = 0 \text{ on the set } N, \tag{3.41}$$

from which and (3.20), $h_t^t = -2\beta$ on N, and consequently $h_t^t = 0$ on N.

Thus, the right hand side of (3.22) becomes $-p^2$, while the left hand side is non-negative on N, and consequently $0 \leqq -p^2$ on N. But, $\beta = 0$, $h_t^t = 0$ and $h_t^t \neq \beta + p$ on N, hence $p\neq 0$ on N. Thus the set N is empty, that is,

$$h_t^t=\beta+p \tag{3.42}$$

on the whole space M.

If we take account of (3.42), (3.22) and (3.24) become

$$\begin{aligned} h_{ji}&=pu_ju_i+\alpha(u_jv_i+v_ju_i)+\beta v_jv_i,\\ k_{ji}&=\alpha u_ju_i+\beta(u_jv_i+v_ju_i)+(k_t^t-\alpha)v_jv_i. \end{aligned} \tag{3.43}$$

Substituting (3.43) into (1.13), we have $K_{kjih}=0$. Therefore, M is locally Euclidean and consequently locally symmetric.

§4. Submanifolds with parallel mean curvature vector

The mean curvature vector of the submanifold M is defined to be

$$\frac{1}{2n}g^{ji}\nabla_jX_i=\frac{1}{2n}h_i^iC+\frac{1}{2n}k_i^iD, \tag{4.1}$$

and the mean curvature H of the submanifold M is defined to be the length of the mean curvature vector, that is,

$$H^2=\frac{1}{4n^2}\{(h_t^t)^2+(k_t^t)^2\}. \tag{4.2}$$

If the mean curvature vector parallel in the normal bundle, then $(h_t^t)^2+(k_t^t)^2=$ const., and

$$k_t^t(\nabla_jl_i-\nabla_il_j)=0,\quad h_t^t(\nabla_jl_i-\nabla_il_j)=0,$$

which is equivalent to

$$\{(h_t^t)^2+(k_t^t)^2\}(\nabla_jl_i-\nabla_il_j)=0, \tag{4.3}$$

that is,

$$\nabla_jl_i-\nabla_il_j=0 \text{ or } (k_t^t)^2+(h_t^t)^2=0$$

on M, that is, the one shows that the normal bundle is trivial, the other that M is minimal.

In §3, we studied the submanifold under the condition $\nabla_jl_i-\nabla_il_j=0$. In this section, we consider the case of which $(h_t^t)^2+(k_t^t)^2=0$, i.e.,

$$h_t^t=0 \text{ and } k_t^t=0. \tag{4.4}$$

Thus, using (2.19,) (2.20) and (4.4), scalar curvature K becomes

$$K=-4(\alpha^2+\beta^2). \tag{4.5}$$

We assume that the scalar curvature K of M is constant. Then, we shall consider two cases, that is, Case I where $K\neq 0$ and Case II where $K=0$.

First we consider Case I where $K\neq 0$. In this case, we first prove

LEMMA 4.1. *Let M be a submanifold of codimension 2 with a complemented f-structure in an even-dimensional Euclidean space such that scalar curvature of M is non-zero constant, and* (2.1) *and* (2.3) *are satisfied. If M is minimal, then the set $N=\{x\in M: \alpha(x)\neq 0\}$ is dense in M.*

Proof. Let V be an open neighborhood such that $V\subset N^c$, where N^c be a complement of N. Since K is non-zero constant, using (4.5), we may assume that $\alpha=0$ and $\beta\neq 0$

on V. Then β is constant on V. Moreover from (2.7),

(4.6) $$k_{jt}v^t=\beta u_j$$

on V. Differentiating (4.6) covariantly, we find

$$(\nabla_k k_{jt})v^t+k_j{}^t(\nabla_k v_t)=\beta(\nabla_k u_j),$$

from which, taking skew-symmetric parts and using (1.15),

$$(l_j h_{kt}-l_k h_{jt})v^t+k_j{}^t(\nabla_k v_t)-k_k{}^t(\nabla_j v_t)=\beta(\nabla_k u_j-\nabla_j u_k),$$

or, using (1.10), (1.11) and (2.6),

$$3\beta(l_j v_k-l_k v_j)-2k_j{}^t k_{ks} f_t{}^s+2\beta h_{kt} f_j{}^t=0,$$

from which, transvecting with u^j,

$$3\beta(l_t u^t)v_k=0,$$

which implies

(4.7) $$l_t u^t=0$$

on V by virtue of the non-zero constance of β. From (2.12), using $\alpha=0$, $\beta\neq 0$, $\beta=$ const. and (4.7), we have

(4.8) $$(2\beta-p)l_j=0.$$

On the other hand, contracting (2.19) with i and j and using $\alpha=0$ and $h_t{}^t=0$, we obtain

(4.9) $$\beta(\beta-p)\geqq 0$$

on V. If β is positive, then from (4.9)

$$2\beta-p\geqq\beta>0, \text{ i.e., } 2\beta-p\neq 0.$$

If β is negative, then from also (4.9)

$$p-2\beta\geqq-\beta>0,$$

therefore, we have

(4.10) $$2\beta-p\neq 0$$

on V. Hence, from (4.8) and (4.10), we find

(4.11) $$l_j=0,$$

which implies $\alpha K=0$ and $\beta K=0$. Thus we have

(4.12) $$\alpha=0 \text{ and } \beta=0$$

on V. This contradicts to $\beta\neq 0$ on V. Hence, any open subset V of N^c is empty. Consequently, Lemma 4.1. is proved.

From the result of Lemma 4.1, in Case I where $K\neq 0$, α is non-zero in the whole space M. Hence, using (2.10) and $h_t{}^t=0$, we have

(4.13) $$\beta+p=0.$$

Substituting (4.13) and $h_t{}^t=0$ into (3.22) and (3.24), we have (3.43). Substituting (3.43) into (1.13), we obtain

(4.14) $$K_{kjih}=(2\alpha^2+\beta^2-\beta p)(u_j v_i u_h v_k+v_j u_i v_h u_k-u_j u_i v_h v_k-v_j v_i u_h u_k).$$

Differentiating (4.14) covariantly and using (1.10), (1.11), (4.13), (3.43) and $h_t^t=0$, we find

$$\nabla_l K_{kjih}=0.$$

Next we consider Case II where $K=0$. From (4.5), we have

(4.15) $$\alpha=0 \text{ and } \beta=0.$$

Moreover, from (3.22) and (4.4), we have

$$0\leqq -p^2$$

that is,

(4.16) $$p=0.$$

Substituting (4.15) and (4.16) into (3.22) and (3.34), respectively, we have

(4.17) $$h_{ji}=0 \text{ and } k_{ji}=0.$$

Thus we have

THEOREM 4.2. *Let M be a submanifold of codimension* 2 *with a complemented f-structure in an even-dimensional Euclidean space E^{2n+2} such that the scalar curvature of M is const., and* (2.1) *and* (2.3) *are satisfied. If M is minimal, then the submanifold is locally symmetric.*

Combining Theorem 3.3 and Theorem 4.2, we obtain

THEOREM 4.3. *Let M be a submanifold of codimension* 2 *with a complemented f-structure in an even-dimensional Euclidean space E^{2n+2} such that the scalar curvature of M is constant, and* (2.1) *and* (2.3) *are satisfied. If the mean curvature vector parallels in the normal bundle, then M is locally symmetric.*

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