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ON SEMI-SIMPLE RINGS AND THEIR COMPLETE MATRIX RINGS

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1. Introduction.

Let R be a ring and M be a right R-module. In this paper we consider the class of all large submodules of M and denote their total intersection by $S(M)$. In section 2, we prove $S(M)$ coincides with the sum of all simple submodules of M, the largest semi-simple submodule in M. Applying this result to an arbitrary ring R whether or not R contains the identity 1, we prove that the complete matrix ring R_n of all $n\times n$ matrices over R is semi-simple if the ring R is semi-simple as a right R-module R_R. This proof is given in Section 3. We also investigate semi-simple right ideals of R and R_n and study their relations.

2. Preliminaries.

We call a submodule P of M *large* in M and write $P\subseteq' M$ in case each non-zero submodule of M meets P. The aim of this section is to prove that $S(M)$ coincides with the sum of all simple submodules of M and to seek a necessary and sufficient condition for a module to be semi-simple.

First, we introduce the definition:

DEFINITION. A submodule N of M is *closed* if and only if N has no proper large extensions in M.

If $M_R\supseteq P_R$, then C is called a *complement submodule of P in M* in case C is a submodule which is maximal in the set of all submodules Q such that $Q\cap P=0$. By Zorn's lemma, if $P\cap A=0$, then there exists a complement submodule of P in M containing A. By a *complement submodule* we mean a submodule which is a complement submodule of some submodule of M. It is easy to see that the closed submodules of a module M coincide with the complement submodules of M. By this fact, P is large in M if and only if P meets every non-zero closed submodule of M. For, if $P\cap K=0$, then we can choose a complement (=closed) submodule C of P containing K. If P meets every non-zero closed submodule, then $C=0$, since $C\cap P=0$ and so $K=0$. This shows that P is large in M. From this we prove the following lemma:

LEMMA 1. *Let A and B be submodules of M. Then B is large in A if and only if there exists a large submodule P of M such that $B=A\cap P$.*

Proof. Assume that $B\subseteq' A$ and K be a complement submodule of B in M. Put $P=B+K$. Since $B\cap(A\cap K)=B\cap K=0$ and $A\cap K=0$, $A\cap P=A\cap(B+K)=B+(A\cap K)=B$. Let D be a submodule of M with $P\cap D=0$. Then also $B\cap(K+D)=B\cap(P\cap(K+D))=B\cap(K+0)=B\cap K=0$. By maximality of K, $D\subseteq K$, hence $D=(B+K)\cap D=0$. Thus

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P is large in M. If P is large in M, then $P\cap A$ is large in A for every submodule A of M. This proves that B is large in A if $B=A\cap P$ where P is large in M.

Let N be any submodule of M. We consider $S(N)$ in N, that is, the intersection of all large submodules of N. Then the following relation holds between $S(N)$ and $S(M)$.

THEOREM 1. $S(N)=S(M)\cap N$.

Proof. By Lemma 1, $\{P\cap N: P\subseteq' M\}=\{Q: Q\subseteq' N\}$ for any submodule N of M. It follows that

$$\begin{aligned} S(M)\cap N &= \cap\{P: P\subseteq' M\}\cap N \\ &= \cap\{P\cap N: P\subseteq' M\} \\ &= \cap\{Q: Q\subset' N\} \\ &= S(N). \end{aligned}$$

Let $f: M\to M'$ be an epimorphism and P' be large in M'. Then $f^{-1}P'\cap A=0$ implies $P'\cap fA=0$ so that $0=fA\subseteq P'$. Thus $A\subseteq f^{-1}fA\subseteq f^{-1}P'\cap A=0$, so $f^{-1}P'$ is large in M. Hence we obtain the following corollary:

COROLLARY 1. (1) *Let f be an R-homomorphism of M into M'. Then $fS(M)\subseteq S(M')$.* (2) *If N is a submodule of M, then $(S(M)+N)/N\subseteq S(M/N)$.*

Proof. (1): Let $y=fx, x\in S(M)$, and let Q be an arbitary large submodule of fM. Since $f^{-1}Q$ is large in $M, x\in f^{-1}Q$ so that $y=fx\in Q$. Hence $fS(M)\subseteq S(fM)\subseteq S(M')$. (2) is an immediate consequence of (1).

We call a module M is *semi-simple* if M is a direct sum of simple submodules. It is the same thing to require that each submodule of M is a direct summand of M [1, p. 55].

COROLLARY 2. *M is semi-simple if and only if $S(M)=M$. Therefore $S(M)$ is the largest semi-simple submodule of M.*

Proof. Assume that M is semi-simple and let A be any non-zero simple submodule of M. Then for each large submodule P of $M, A\cap P\neq 0$ so that $A=A\cap P\subseteq P$. Thus $A\subseteq S(M)$ and $M=S(M)$. Conversely, if A is a submodule of M and B is any complement submodule of A in M, then $A\oplus B$ is large in M and $S(M)=M$ implies $S(M)\subseteq A\oplus B=M$ so that A is a direct summand of M. Hence M is semi-simple. By Theorem 1, $S(S(M))=S(M)\cap S(M)=S(M)$ and $S(M)$ is semi-simple by the above result. If a submodule P is semi-simple, then $P=S(P)=P\cap S(M)\subseteq S(M)$. Therefore $S(M)$ is a semi-simple submodule of M which contains every semi-simple submodule.

Immediately, we have:

COROLLARY 3. *The total intersection $S(M)$ of all large submodules of M is the sum of all simple submodules of M.*

It is easy to give an example for $S(M/S(M))\neq 0$. But under some conditions we can get $S(M/S(M))=0$. If $M=S(M)$, it is clear. Now assume that $M\neq S(M)$ and we prove $S(M/S(M))=0$ if $S(M)$ is closed in M. Let $\bar{P}$ be a simple submodule of $\bar{M}=M/S(M)$. Since there is a 1:1 correspondence between submodules of $\bar{M}$ and submodules of M containing $S(M)$, either $P=S(M)$ or there are no submodules between P and $S(M)$

where P is an inverse image of $\bar{P}$ by a projection map. If $S(M)$ is not large in P, then, since $S(P)=P\cap S(M)=S(M)$, P is the only submodule which is large in P, contradicting to $S(P)=S(M)$. So $S(M)$ is large in P. Thus we have the following:

COROLLARY 4. *If M is a module in which $S(M)$ is closed, then $S(M/S(M))=0$.*

3. Semi-simple rings.

We now turn our attention to a ring R regarded as right R-module R_R. We call a right ideal K (hence a right R-module) of R simple in case the only right ideals of R contained in K are 0 and K itself; K is semi-simple if it is the sum of simple right ideals. In this section we characterize simple right ideals and semi-simple right ideals of a ring R with the identity 1 and of the complete matrix ring R_n of all $n\times n$ matrices over R. Using these results and applying the results obtained in Section 2, we prove that for any ring R (whether or not R contains 1) $S(R_n)=(S(R))_n$ and also prove that if a ring R is semi-simple as a right R-module R_R, then so is its complete matrix ring R_n. First we consider a ring R with the identity 1. To avoid the complexity we employ the following notations: For each right ideal K of R, and each $p=1,2,\cdots,n$, write

$$K_n^{(p)}=\{A=(a_{ij})\in R_n : a_{ij}=0 \text{ if } i\neq p,\ a_{pj}\in K,\ j=1,2,\cdots,n.\}$$

and for each right ideal $\boldsymbol{K}$ of R_n, and each p, put $K_{(p)}$ as follows:

$$K_{(p)}=\{a\in R : a=a_{p1} \text{ for some } A=(a_{ij}) \text{ in } \boldsymbol{K}\}.$$

First, we prove that $K_n^{(p)}$ and $K_{(p)}$ are right ideals of R_n and R respectively.

LEMMA 2. *For each $p=1,2,\cdots,n$, $K_n^{(p)}$ and $K_{(p)}$ are right ideals of R_n and R respectively. Furthermore $K_n=\sum_{p=1}^{n}K_n^{(p)}$ and $\boldsymbol{K}\subseteq\sum_{p=1}^{n}(K_{(p)})_n$.*

Proof. We denote the matrix units of R_n by E_{ij}. Let $A=(a_{ij})$ and $B=(b_{ij})$ in $K_n^{(p)}$ and $C=(c_{ij})$ be an arbitrary element of R_n. Then $A-B=(a_{ij}-b_{ij})$ and $a_{ij}-b_{ij}=0$ if $i\neq p$ and $a_{pj}-b_{pj}\in K$ for each j, so that $K_n^{(p)}$ is closed under subtraction. For each $r,s=1,2,\cdots,n$, $A(c_{rs}E_{rs})=(\sum_{i,j}a_{ij}E_{ij})(c_{rs}E_{rs})=\sum_{i}a_{ir}c_{rs}E_{is}$ is a matrix whose i-th rows are all zero if $i\neq p$ and $a_{pr}c_{rs}$ in K. But AC is a sum of such matrices, and therefore $AC\in K_n^{(p)}$. This proves $K_n^{(p)}$ is a right ideal in R_n. Furthermore, it is easy to check $K_n=\sum_{p=1}^{n}K_n^{(p)}$. Next we will show that $K_{(p)}$ is a right ideal in R $(p=1,2,\cdots,n.)$ and $\boldsymbol{K}\subseteq\sum_{p=1}^{n}(K_{(p)})_n$. Since K is closed under addition (and subtraction), the same is true for $K_{(p)}$. Let a in $K_{(p)}$ and $r\in R$. Then by definition of $K_{(p)}$, there exists a matrix $A=\sum a_{ij}E_{ij}$ in $\boldsymbol{K}$ with $a_{p1}=a$. Since a matrix $A(rE_{11})=\sum_{i}a_{i1}rE_{i1}$ is in $\boldsymbol{K}$ and its $(p,1)$-position element is $a_{p1}r=ar$, $ar\in K_{(p)}$. Thus $K_{(p)}$ is a right ideal in R. Let $A=\sum a_{ij}E_{ij}$ be any element of $\boldsymbol{K}$. Then for any $q=1,2,\cdots,n$, $B=AE_{q1}=(\sum_{i,j}a_{ij}E_{ij})E_{q1}=\sum_{i}a_{iq}E_{i1}$ is a matrix in $\boldsymbol{K}$ whose $(p,1)$-position element is a_{pq}. This is true for each $p=1,2,\cdots,n$, and therefore $a_{pq}\in K_{(p)}$ for each q. Since $a_{pq}E_{pq}\in(K_{(p)})_n$ and $A=\sum_{p,q}a_{pq}E_{pq}\in\sum_{p=1}^{n}(K_{(p)})_n$, it follows that $\boldsymbol{K}$ is contained in $\sum_{p=1}^{n}(K_{(p)})_n$. This completes the proof of lemma.

THEOREM 2. *If K is a simple right ideal of R, then $K(p)$ is a simple right ideal of R_n for each $p=1,2,\cdots,n$, and therefore K_n is semi-simple in R_n.*

Proof. Let $\boldsymbol{N}$ be a right ideal of R_n such that $\boldsymbol{N}\subseteq K_n^{(p)}$. Then $N_{(p)}$ is a right ideal of R satisfying $(N_{(p)})_n^{(p)}=\boldsymbol{N}$. For, if $A=(a_{ij})\in\boldsymbol{N}$, then $AE_{j1}=\sum_i a_{ij}E_{i1}=a_{pj}E_{p1}\in\boldsymbol{N}$ and $a_{pj}\in N_{(p)}$ for each j. It follows that $A\in(N_{(p)})_n^{(p)}$ and hence $\boldsymbol{N}\subseteq(N_{(p)})_n^{(p)}$. Suppose now that $A\in(N_{(p)})_n^{(p)}$ and let us show that $A\in\boldsymbol{N}$. Let $a=a_{pj}$ be an element in the (p,j)-position of A. Then there exists a matrix $B=(b_{ij})\in\boldsymbol{N}$ with $b_{p1}=a$. Since $BE_{1j}=\sum_i b_{i1}E_{ij}=b_{p1}E_{pj}=aE_{pj}\in\boldsymbol{N}$, $A=\sum_{j=1}^{n}a_{pj}E_{pj}\in\boldsymbol{N}$ and so $(N_{(p)})_n^{(p)}=\boldsymbol{N}$. Since $\boldsymbol{N}\subseteq K_n^{(p)}$, $N_{(p)}\subseteq K$, and since K is simple, either $N_{(p)}=0$ or $N_{(p)}=K$. i. e., $\boldsymbol{N}=0$ or $\boldsymbol{N}=K_n^{(p)}$. This proves that $K_n^{(p)}$ is simple and since $K_n=\sum_{p=1}^{n}\oplus K_n^{(p)}$ is a direct sum of simple right ideals, K_n is semi-simple.

Now the following lemma can be proved straightforwardly, so the proof will be omitted.

LEMMA 3. *If $K=\sum_{i\in I}K_i$ is a sum of right ideals of R, then $K_n^{(p)}=(\sum_{i\in I}K_i)_n^{(p)}=\sum_{i\in I}(K_i)_n^{(p)}$.*

COROLLARY 5. *If K is semi-simple in R, then so is $K_n^{(p)}$ for each $p=1,2,\cdots,n$.*

Proof. Write $K=\sum_{i\in I}K_i$ where K_i is simple in R. Then by Theorem 2, for each $i\in I$, $(K_i)_n^{(p)}$ is a simple right ideal of R_n. Since $K_n^{(p)}=\sum_{i\in I}(K_i)_n^{(p)}$ is a sum of simple right ideals, $K_n^{(p)}$ is semi-simple for each $p=1,2,\cdots,n$.

Since, for each right ideal K of R, we have $K_n=\sum_{p=1}^{n}K_n^{(p)}$, we obtain the following corollary:

COROLLARY 6. *If K is semi-simple in R, then so is K_n in R_n.*

THEOREM 3. *If R is a ring with the identity* 1, *then $(S(R))_n$ is semi-simple in R_n.*

Proof. Write $S(R)=\sum_{i\in I}K_i$ where K_i are simple right ideals of R. Then $(S(R))_n^{(p)}=\sum_{i\in I}(K_i)_n^{(p)}$ and each $(K_i)_n^{(p)}$ is simple by Theorem 2, so that $(S(R))_n^{(p)}$ is semi-simple. But $(S(R))_n=\sum_{i\in I}(S(R))_n^{(p)}$ is a sum of semi-simple right ideals in R_n, and therefore $(S(R))_n$ is semi-simple.

We know that $S(M)$ is the largest semi-simple submodule of M by Corollary 2. Therefore $(S(R))_n$ is contained in $S(R_n)$ by the above result. To prove the converse inclusion, we need the following lemma:

LEMMA 4. *If $\boldsymbol{K}$ is a simple (resp. large) right ideal of R_n, then there exists a semi-simple (resp. large) right ideal K of R such that $\boldsymbol{K}\subseteq K_n$.*

Proof. Consider a right ideal $K_{(p)}=\{a\in R: a=a_{p1}$ for some $A=(a_{ij})\in\boldsymbol{K}\}$ and let $K=$

$\sum_{p=1}^{n} K_{(p)}$. Then by Lemma 2, K is a right ideal of R such that $K \subseteq K_n$. First assume that $\boldsymbol{K}$ is simple and we show that $K_{(p)}$ is simple in R. For this purpose, let $N_{(p)}$ be a right ideal of R such that $N_{(p)} \subseteq K_{(p)}$ and let $\boldsymbol{N} = (N_{(p)})_n^{(p)} + \sum_{i \neq p} R_n^{(i)}$, that is, any matrix $A = (a_{ij})$ in $\boldsymbol{N}$ is of the form: for each $j = 1, 2, \cdots, n$, $a_{pj} \in N_{(p)}$ and if $i \neq p$, then a_{ij} is an arbitary element of R. We note that $\boldsymbol{N} \cap \boldsymbol{K} = \{A \in K : A = (a_{ij}),\ a_{pj} \in N_{(p)}$ for each $j\}$. Since $\boldsymbol{K}$ is simple, it follows that $\boldsymbol{N} \cap \boldsymbol{K}$ is $\boldsymbol{K}$ or 0 and so $N_{(p)} = K_{(p)}$ or $N_{(p)} = 0$, that is, $K_{(p)}$ is simple for each $p = 1, 2, \cdots, n$. Thus K is a semi-simple right ideal of R such that $K \subseteq K_n$. If $\boldsymbol{K}$ is large in R_n, then K is also large in R since $\boldsymbol{K} \subseteq K_n$. For, if P is a right ideal of R such that $K \cap P = 0$, then $(K \cap P)_n = K_n \cap P_n = 0$ so that $P_n = 0$ and $P = 0$. This completes the proof of lemma.

The following result is an immediate consequence of Lemma 4 and Theorem 3.

COROLLARY 7. *If R is a ring with the identity* 1, then $S(R_n) = (S(R))_n$.

Now we prove the following theorem which is a generalization of the above result.

TEHOREM 4. *For any ring R, $S(R_n) = (S(R))_n$.*

Proof. If $1 \in R$, then it is through. If $1 \notin R$, then we imbed R into the ring R' with the identity 1 as an ideal and by the case already proved we have $S(R'_n) = (S(R'))_n$. Theorem 1 then shows that $S(R) = S(R') \cap R$. Since R_n is an ideal in R'_n, we can again apply Theorem 1 and obtain

$$S(R_n) = R_n \cap S(R'_n) = R_n \cap (S(R'))_n = (R \cap S(R'))_n = (S(R))_n.$$

This completes the proof of the theorem.

By the above theorem, we can prove the following theorem which is the main result of this section.

THEOREM 5. *If a ring R is semi-simple as a right R-module R_R, then so is R_n.*

Proof. Theorem 4 ensures that $S(R_n) = (S(R))_n = R_n$ if R is semi-simple. Therefore R_n is also semi-simple by Corollary 2.

References

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