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ON PROJECTIVE LIMITS IN PRIMITIVE CATEGORIES

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1. Introduction

Let $\mathbf{C}$ be a category and let $\otimes$ be a functor from the product category $\mathbf{C}\times\mathbf{C}$ into $\mathbf{C}$ such that the functors

$$\otimes(\otimes\times\mathbf{C}):\ (\mathbf{C}\times\mathbf{C})\times\mathbf{C}\to\mathbf{C}\times\mathbf{C}\to\mathbf{C}$$

and

$$\otimes(\mathbf{C}\times\otimes):\ \mathbf{C}\times(\mathbf{C}\times\mathbf{C})\to\mathbf{C}\times\mathbf{C}\to\mathbf{C}$$

are identical and I be an object of $\mathbf{C}$ such that the functors obtained by $\otimes$, fixing the variables in I, are identical to $1_{\mathbf{C}}$. Then a triple $(\mathbf{C},\otimes,I)$ is called a multipicative category of $\mathbf{C}$.

For a multiplicative category $(\mathbf{C},\otimes,I)$ of $\mathbf{C}$, a triple (A,m,k) is called a tensor object or $\otimes$-object of $\mathbf{C}$, where A is an object of $\mathbf{C}$, m is a morphism from $A\otimes A$ to A such that $m(m\otimes A)=m(A\otimes m)$ and k is a morphism from I to A such that $m(k\otimes A)=$ $=m(A\otimes k)=1_A$.

If we now define morphisms f from (A,m,k) to (A',m',k') as follows; for $f:A\to A'$,

$$f\cdot m=m'\cdot(f\otimes f),\qquad k'=f\cdot k$$

and their composition is the composition in $\mathbf{C}$, then the class of objects, which is the class of all $\otimes$-objects of $\mathbf{C}$, together with the class of morphisms f: $(A,m,k)\to(A',m',k')$, and the composition in $\mathbf{C}$ forms a category. This category is called a primitive category and we denote it by $\Phi(\mathbf{C},\otimes,I)$. The functor $\psi:\Phi(\mathbf{C},\otimes,I)\to\mathbf{C}$ is forgetful if for every object (A,m,k) and f: $(A,m,k)\to(A',m',k')$, $\psi[(A,m,k)]=A$, $\psi(f)=f$ (For other equivalent definitions see [1], [3]). The notion of multiplicative category was established by J. Bénabou [2] and the structure of the primitive categories was studied by B. Eckmann and P. J. Hilton [4].

The purpose of this note is to establisih the simple results on the projectve limits in primitive categories.

2. Primitive categories

Let $(\mathbf{C},\otimes,I)$ and $(\bar{\mathbf{C}},\otimes,\bar{I})$ be two multiplicative categories of $\mathbf{C}$ and $\bar{\mathbf{C}}$ respectively. A triple (G,φ,δ) is called a morphism from $(\mathbf{C},\otimes,I)$ to $(\bar{\mathbf{C}},\otimes,\bar{I})$, where G is a functor from $\mathbf{C}$ into $\bar{\mathbf{C}}$, φ is a natural transformation from $\otimes(G\times G)$ to $G\otimes$ such that for all triple (A,B,C) of objects of $\mathbf{C}$,

$$\varphi_{(A\otimes B,C)}\cdot(\varphi_{(A,B)}\otimes G(C))=\varphi_{(A,B\otimes C)}\cdot(G(A)\otimes\varphi_{(B,C)})$$

and $\delta:\bar{I}\to G(I)$ (in $\bar{\mathbf{C}}$) is a morphism such that for all objects $A\in\mathbf{C}$,

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$$\varphi_{(I,A)}\cdot(\bar{\delta}\otimes G(A))=1_{G(A)}=\varphi_{(A,I)}\cdot(G(A)\times\delta).$$

For two morphisms $(G,\varphi,\delta):(\mathbf{C},\otimes,I)\to(\bar{\mathbf{C}},\otimes,\bar{I})$ and $(\bar{G},\bar{\varphi},\bar{\delta}):(\bar{\mathbf{C}}\otimes,\bar{I})\to(\bar{\bar{\mathbf{C}}},\otimes,\bar{\bar{I}})$, we shall define the composition of (G,φ,δ) and $(\bar{G},\bar{\varphi},\bar{\delta})$ as follows;

$$(\bar{G}\cdot G,(\bar{G}*\varphi)(\bar{\varphi}*(G\times G)),\bar{G}(\delta)\cdot\bar{\delta})$$

where,

$$(G*\varphi)(\bar{\varphi}*(G\times G)):[\otimes(\bar{G}\times\bar{G})](G\times G)=\otimes(\bar{G}G\times\bar{G}G)\to[\bar{G}\otimes](G\times G)$$
$$\to\bar{G}(G\otimes).$$

THEOREM 1. *Let* $(G,\varphi,\delta):(\mathbf{C},\otimes,I)\to(\bar{\mathbf{C}},\otimes,\bar{I})$ *and* $\psi:\Phi(\mathbf{C},\otimes,I)\to\mathbf{C}$ *and* $\psi':\Phi(\bar{\mathbf{C}},\otimes,\bar{I})\to\bar{\mathbf{C}}$ *be forgetful functors. If for a morphism* (G,φ,δ), *every* $\otimes$*-object* (A,m,k) *and* $f:(A,m,k)\to(A',m',k')$,

$$\Phi(G,\varphi,\delta)[(A,m,k)]=(G(A),G(m)\varphi_{(A,A)},G(k)\delta)\in\Phi(\bar{\mathbf{C}},\otimes,\bar{I})$$

and

$$\Phi(G,\varphi,\delta)[f]=G(f):\ (G(A),G(m)\varphi_{(A,A)},G(k)\delta)$$
$$\to(G(A'),G(m')\varphi_{(A',A)},G(k')\delta)\ (\in\Phi(\bar{\mathbf{C}},\otimes,\bar{I})),$$

then $\Phi(G,\varphi,\delta)$ *is a functor from* $\Phi(\mathbf{C},\otimes,I)$ *into* $\Phi(\bar{\mathbf{C}},\otimes,\bar{I})$ *such that* $G\psi=\psi'\Phi(G,\varphi,\delta)$. *And* $\Phi(G,\varphi,\delta)$ *is a faithful functor if and only if so is* G.

Proof. For two morphisms $f:(A,m,k)\to(A',m',k')$ and $f':(A',m',k')\to(A'',m'',k'')$ in $\Phi(\mathbf{C},\otimes,I)$,

$$\Phi(G,\varphi,\delta)\ [f'\cdot f]=G[f'\cdot f]=G[f']\cdot G[f]$$
$$=\Phi(G,\varphi,\delta)[f']\cdot\Phi(G,\varphi,\delta)[f].$$
$$\Phi(G,\varphi,\delta)\ [1_A]=G[1_A]=1_{G(A)}=1_{\Phi(G,\varphi,\delta)[(A,m,k)]},$$

and

$$G\cdot\psi[(A,m,k)]=G[A],$$
$$\psi'\cdot\Phi(G,\varphi,\delta)[(A,m,k)]=\psi'(G(A),G(m)\varphi_{(A,A)},G(k)\delta]=G(A)$$

Hence

$$G\psi=\psi'\Phi(G,\varphi,\delta).$$

Since the function

$$\chi:\mathrm{Hom}_{\Phi(\mathbf{C},\otimes,I)}[(A,m,k),(A',m',k')]\longrightarrow$$
$$\mathrm{Hom}_{\Phi(\bar{\mathbf{C}},\otimes,\bar{I})}[(G(A),G(m)\varphi_{(A,A)},G(k)\delta),(G(A'),G(m')\varphi_{(A',A')},G(k')\delta)]$$

is univalent iff χ_0: $\mathrm{Hom}_{\mathbf{C}}(A,A')\to\mathrm{Hom}_{\bar{\mathbf{C}}}(G(A),\ G(A'))$ is univalent, $\Phi(G,\varphi,\delta)$ is faithful iff so is G.

DEFINITION. For a morphism $(G,\varphi,\delta):(\mathbf{C},\otimes,I)\to(\bar{\mathbf{C}},\otimes,\bar{I})$, if $G\otimes=\otimes(G\times G)$ and $\bar{I}=G(I)$, i.e. $\varphi=1_{\otimes}$, $\delta=1_{\bar{I}}$ then we say that the functor G commutes to tensor product [1].

In this case we denote the morphism $(G, 1_{\otimes}, 1_I)$ by G and $\Phi(G, 1_{\otimes}, 1_I) = \Phi(G)$. Then we have

$$\Phi(G)[(A, m, k)] = (G(A), G(m), G(k))).$$

For a functor G: $\mathbf{C} \to \bar{\mathbf{C}}$, a morphism $T: G(A) \to \bar{A}$ in $\mathbf{C}$ is called a universal morphism if for all morphim f: $G(A') \to \bar{A}$ there is a unique morphism $g: A' \to A$ such that $f = T \cdot G(g)$.

LEMMA. *Let* $(\mathbf{C}, \otimes, I)$ *and* $(\bar{\mathbf{C}}, \bar{\otimes}, \bar{I})$ *be two multiplicative categories and* $G: \mathbf{C} \to \bar{\mathbf{C}}$ *be a functor that is commutative to tensor product. Then if for a* $\bar{\otimes}$*-object* $(\bar{A}, \bar{m}, \bar{k})$ of $\Phi(\bar{\mathbf{C}}, \bar{\otimes}, \bar{I})$ *the morphism* $T: G(A) \to \bar{A}$ *in* $\bar{\mathbf{C}}$ *is universal, there exist unique morphisms* $m: A \otimes A \to A$ and k: $I \to A$ *such that* $(A, m, k) \in \Phi(\mathbf{C}, \otimes, I)$ *and* T: $\Phi(G)[(A, m, k)] \to (\bar{A}, \bar{m}, \bar{k})]$ *is universal.*

Proof. Since T is a universal morphism, for a morphism f: $G(A \otimes A) \to \bar{A}$ there is a unique morphism $m: A \otimes A \to A$ such that

$$\begin{array}{ccc} G(A) & \xrightarrow{\quad T \quad} & \bar{A} \\ \uparrow{\scriptstyle G(m)} & \nearrow{\scriptstyle f} & \\ G(A \otimes A) & & \end{array}$$

commutes. Similarly, there is a unique k: $I \to A$ with $T \cdot G(k) = \bar{k}$. By the uniqueness, $m \otimes (m \otimes A) = m \otimes (A \otimes m)$ and $m(A \otimes k) = m(k \otimes A) = 1_A$. If T is universal then by the definition of the morphism $T: (G(A), G(m), G(k)) \to (\bar{A}, \bar{m}, \bar{k})$ we can easily know that the latter is also universal.

3. Projective limits

For any two categories $\mathbf{A}$ and $\mathbf{B}$, let $\mathbf{Cat}(\mathbf{A}, \mathbf{B})$ denote the category of all covariant functors from $\mathbf{A}$ into $\mathbf{B}$. Let $K_A: \mathbf{J} \to \mathbf{C}$ be the constant functor which assigns each object of $\mathbf{J}$ to the fixed object A of $\mathbf{C}$ and each morphism of $\mathbf{J}$ to 1_A and let $E_{\mathbf{J}}$: $\mathbf{C} \to \mathbf{Cat}(\mathbf{J}, \mathbf{C})$ be a functor such that $E_J(A) = K_A$ for all $A \in \mathbf{C}$. For any functor $F \in \mathbf{Cat}(\mathbf{J}, \mathbf{C})$ a natural transformation ρ: $E_{\mathbf{J}}(A) = K_A \to F$ is called the projective limit of F if for any natural transformation σ: $E_{\mathbf{J}}(A') \to F$ there exist the unique morphism μ: $A' \to A$ in $\mathbf{C}$ such that

$$\sigma = \rho \cdot E_J(\mu).$$

Now we suppose that for any two objects (A, m, k) and (A', m', k') of $\Phi(\mathbf{C}, \otimes, I)$,

$$(A \otimes A') \otimes (A \otimes A') = (A \otimes A) \otimes (A' \otimes A').$$

Let us denote a morphism from $(A \otimes A') \otimes (A \otimes A')$ to $A \otimes A'$ by $m \otimes m'$ and a morphism for I to $A \otimes A'$ by $k \otimes k'$. Then the morphism $m \otimes m'$ is associative and

$$\begin{aligned} (m \otimes m')[(k \otimes k') \otimes (A \otimes A')] &= [m(k \otimes A)] \otimes [m'(k' \otimes A')] \\ &= 1_A \otimes 1_{A'} \\ &= 1_{A \otimes A'}. \end{aligned}$$

Similarly,

$$(m\otimes m')[(A\otimes A')\otimes(k\otimes k')]=1_{A\otimes A'}.$$

Hence, in this case we have a functor

$$\odot: \Phi(\mathbf{C},\otimes,I)\times\Phi(\mathbf{C},\otimes,I)\to\Phi(\mathbf{C},\otimes,I)$$

such that

$$\odot[(A,m,k),\ (A',m',k')]=(A,m,k,)\odot(A',m',k')=(A\otimes A',m\otimes m',k\otimes k')$$

and for f: $(A,m,k)\to(B,n,h)$ and f': $(A',m',k')\to(B',n',h')$

$$\odot[(f,f')]=f\otimes f': (A\otimes A',m\otimes m',k\otimes k')\to(B\otimes B',n\otimes n',h\otimes h').$$

The functor $\odot$ is associative and

$$(A,m,k)\odot(I,1_I,1_I)=(A,m,k).$$

Therefore given a multipicative category $(\mathbf{C},\otimes,I)$ we can also define a multiplicative category $\{\Phi(\mathbf{C},\otimes,I),\odot,\mathbf{I}=(I,1_I,1_I)\}$ of a primitive category $\Phi(\mathbf{C},\otimes,I)$. And if

$$\mathbf{Cat}(\mathbf{J},\Phi(\mathbf{C},\otimes,I))\times\mathbf{Cat}(\mathbf{J},\Phi(\mathbf{C},\otimes,I))=\mathbf{Cat}(\mathbf{J},\Phi(\mathbf{C},\otimes,I)\times\Phi(\mathbf{C},\otimes,I)),$$

then we can define a functor

$$\mathbf{Cat}(\mathbf{J},\odot): \mathbf{Cat}(\mathbf{J},\Phi(\mathbf{C},\otimes,I)\times\mathbf{Cat}(\mathbf{J},\Phi(\mathbf{C},\otimes,I))\to\mathbf{Cat}(\mathbf{J},\Phi(\mathbf{C},\otimes,I))$$

as follows:

$$\mathbf{Cat}(\mathbf{J},\odot)[(\phi,\psi)]=\phi\odot\psi,$$

$$[\phi\odot\psi](j)=\phi(j)\odot\psi(j),$$

for all $j\in\mathbf{J}$. Since the functor $\odot$ is associative, so is the functor $\mathbf{Cat}(\mathbf{J},\ \odot)$. Let us denote $\mathbf{Cat}(\mathbf{J},\odot)$ by $\odot$.

If $E_{\mathbf{J}}$: $\Phi(\mathbf{C},\otimes,I)\to\mathbf{Cat}(\mathbf{J},\Phi(\mathbf{C},\otimes,I)$ be a functor such that for each object (A,m,k) of $\Phi(\mathbf{C},\otimes,I)$ and for all objects j of $\mathbf{J}$

$$[E_{\mathbf{J}}(A,m,k)](j)=(A,m,k),$$

then for $E_{\mathbf{J}}(I,1_I,1_I)=\bar{\mathbf{I}}$ and $\phi\in\ \mathbf{Cat}(\mathbf{J},\Phi(\mathbf{C},\otimes,I)$ and $\phi(j)=(B,n,h)$,

$$(\bar{\mathbf{I}}\odot\phi)(j)=(I\otimes B,1_I\otimes n,1_I\otimes h)=(B,n,h)=\phi(j).$$

Similarly, $\phi\odot\bar{\mathbf{I}}=\phi$. Hence we have a multiplicative category $[\mathbf{Cat}(\mathbf{J},\Phi(\mathbf{C},\otimes,I),\odot,\bar{\mathbf{I}}]$ of a category $\mathbf{Cat}(\mathbf{J},\Phi(\mathbf{C},\otimes,I)$.

For each ordered pairs $\{(A,m,k),(A',m',k')\}$ of objects of $\Phi(\mathbf{C},\otimes,I)$ and for each $j\in\mathbf{J}$,

$$\begin{aligned}\odot\cdot E_{\mathbf{J}}\times E_{\mathbf{J}}[\{(A,m,k),(A',m',k')\}](j)&=[E_{\mathbf{J}}(A,m,k)\odot E_{\mathbf{J}}(A',m',k')](j)\\&=E_{\mathbf{J}}(A,m,k)(j)\odot E_{\mathbf{J}}(A',m',k')(j)\\&=(A,m,k)\odot(A',m',k')\\&=(A\otimes A',m\otimes m',k\otimes k').\end{aligned}$$

Similarly,

$$E_{\mathbf{J}}\odot[\{(A,m,k),\ (A',m',k')\}](j)=(A\otimes A',\ m\otimes m',\ k\otimes k').$$

The functor $\odot$ commutes to the tensor product.

Let us denote

$$(E_{\mathbf{J}}, 1_{\odot}, 1_{\bar{\mathbf{I}}}):[\Phi(\bar{\mathbf{C}},\otimes,I),\odot,\ \ (I,1_I,1_I)]\rightarrow[\mathbf{Cat}(\mathbf{J},\Phi(\mathbf{C},\otimes,I)),\odot,\bar{\mathbf{I}}]$$

by $E_{\mathbf{J}}$. Then we have a functor

$$\Phi(E_{\mathbf{J}}):\Phi[\Phi(\mathbf{C},\otimes,I),\odot,(I,1_I,1_I)]\rightarrow\Phi[\mathbf{Cat}(\mathbf{J},\Phi(\mathbf{C},\otimes,I,)),\odot,\bar{\mathbf{I}}]$$

such that for each $[(A,m,k).\bar{m},\bar{k}]\in\Phi[\Phi(\mathbf{C},\otimes,I),\odot,(I,1_I,1_I)$, where

$$\bar{m}:(A,m,k)\odot(A,m,k)\rightarrow(A,m,k) \text{ and } \bar{k}:\ (I,1_I,1_I)\rightarrow(A,m,k).$$

Hence we have the

THEOREM 2. *Let* $(\mathbf{C},\otimes,I)$ *be a multiplicative category of* $\mathbf{C}$ *and* $(A\otimes A')\otimes(A\otimes A')=(A\otimes A)\otimes(A'\otimes A')$. *Then for* $\phi\in\mathbf{Cat}(\mathbf{J},\Phi(\mathbf{C},\otimes,I))$ *and* $(\phi,m_\phi,k_\phi)\in\Phi[\mathbf{Cat}(\mathbf{J},\Phi(\mathbf{C},\otimes,I)),\odot,\bar{I}]$, *the functor* ϕ *has a projective limit* ρ: $E_J[(A,m,k)]\rightarrow\phi$ *if and only if* $\Phi(\rho):\Phi(E_{\mathbf{J}})[(A,m,k),\bar{m},\bar{k}]\rightarrow(\phi,m_\phi,k_\phi)$ is universal.

Proof. Let $\Phi(\rho)$ be universal, then by the definition of $\Phi(\rho)$, ρ is universal. Conversely, if φ has a projective limitρ, then by the Lemma $\Phi(\rho)$ is universal.
only if ρ is universal.

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