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TOPOLOGICAL SPACES WITH SEMIDEVELOPMENTS

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The principal results of the paper are as follows. Every cushioned pair-semidevelopable space is regular. A locally semidevelopable space is semidevelopable if and only if it is subparacompact. A locally cushioned pair-semidevelopable space is cushioned pair-semidevelopable if it is subparacompact and collectionwise normal.

A topological space X is said to be *semidevelopable* [1] if there is a sequence of (not necessarily open) covers of X, $\gamma=\{\gamma_n\}_{n=1}^{\infty}$ such that for each $x\in X$, $\{\mathrm{St}(x,\gamma_n)\}_{n=1}^{\infty}$ is a neighborhood base at x. In this case, γ is called a *semidevelopment* for X.

If γ and δ are collections of subsets of X, then we say that γ is cushioned in δ if one can assign to each $G\in\gamma$ a $D(G)\in\delta$ such that, for every $\gamma'\subset\gamma$,

$$\mathrm{Cl}(\bigcup\{G|G\in\gamma'\})\subset\bigcup\{D(G)|G\in\gamma'\}.$$

By a *cushioned pair-semidevelopment* [2] for X we shall mean a pair of semidevelopments (γ,δ) such that γ_n is cushioned in δ_n for each n. A topological space X is said to be *cushioned pair-semidevelopable* if there exists a cushioned pair-semidevelopment for X. Unless otherwise stated no separation axioms are assumed.

THEOREM 1. *Every cushioned pair-semidevelopable space is regular.*

Proof. Let (γ,δ) be a cushioned pair-semidevelopment of X. For each $x\in X$, let U be an open set containing the point x. Since δ is a semi-development, there is an integer m such that $x\in\mathrm{St}(x,\delta_m)\subset U$. For such m, we put $\gamma_m'=\{G|x\in G\in\gamma_m\}$. Since γ_m is cushioned in δ_m,

$$\begin{aligned}x\in\mathrm{Int}\,\mathrm{St}(x,\gamma_m)\subset\mathrm{Cl}\,\mathrm{St}(x,\gamma_m)&=\mathrm{Cl}(\bigcup\{G|G\in\gamma_m'\})\\&\subset\bigcup\{D(G)|G\in\gamma_m'\}\subset\mathrm{St}(x,\delta_m)\subset U.\end{aligned}$$

A space is *subparacompact* [5] if every open cover has a σ-discrete closed refinement. A space is *collectionwise normal* [3] if for every discrete collection of subsets $\{H_\alpha|\alpha\in A\}$ there is a discrete collection of open subsets $\{G_\alpha|\alpha\in A\}$ such that $H_\alpha\subset G_\alpha$ for every $\alpha\in A$.

Smirnov [7] has shown that a locally metric space is metrizable if it is paracompact. Ceder [6] has obtained that a locally stratifiable T_1-space is a stratifiable T_1-space if it is paracompact. We can obtain the following

THEOREM 2. *A locally cushioned pair-semidevelopable space X is cushioned pair-semidevelopable if it is subparacompact and collectionwise normal.*

Proof. For each $x\in X$, there is an open neighborhood U_x of x with a cushioned pair-semidevelopment. Since X is subparacompact, there is a σ-discrete closed refinement $\mathcal{L}=\bigcup_{n=1}^{\infty}\mathcal{L}_n$ of $\{U_x|x\in X\}$. Now let n be a fixed positive integer. For each $B\in\mathcal{L}_n$, let $x(B)$ be a fixed element of X such that $B\subset U_{x(B)}$. Since X is collectionwise normal,

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the discrete collection $\mathscr{L}_n$ has a discrete collection $\{G(B)\mid B\subset G(B),\ B\in\mathscr{L}_n\}$ of open subsets, and there exist open sets $V'_{x(B)}$ and $V_{x(B)}$ in X such that $B\subset V'_{x(B)}\subset \mathrm{Cl}\,V_x'_{(B)}\subset V_{x(B)}\subset \mathrm{Cl}\,V_{x(B)}\subset (G(B)\cap U_{x(B)})$. Since every cushioned pair-semidevelopable space is hereditarily cushioned pair-semidevelopable, $\mathrm{Cl}\,V_{x(B)}$ has a cushioned pair-semidevelopment $(\gamma(x(B)),\ \delta(x(B)))$. For each $n, m\in N$, we put

$$\gamma_{n,m}=\{G\mid G\in\gamma_m(x(B)), B\in\mathscr{L}_n\}\cup\{Q_n\}$$

and

$$\delta_{n,m}=\{G\mid G\in\delta_m(x(B)), B\in\mathscr{L}_n\}\cup\{\mathrm{Cl}\,Q_n\}$$

where $Q_n=X-\cup\{\mathrm{Cl}\,V'_{x(B)}\mid B\in\mathscr{L}_n\}$. Then $\gamma=\{\gamma_{n,m}\mid n, m\in N\}$ and $\delta=\{\delta_{n,m}\mid n, m\in N\}$ are sequences of covers of X and we show that (γ, δ) is a cushioned pairsemidevelopment.

For each $z\in X$, there is an integer $n\in N$ such that $z\in B\in\mathscr{L}_n$. If U is any open set containing z, there exists some $m\in N$ such that $z\in \mathrm{Int}_{(\mathrm{Cl}\,V_{x(B)})}\mathrm{St}(z, \gamma_m(x(B)))\subset \mathrm{St}(z, \gamma_m(x(B)))\subset (U\cap \mathrm{Cl}\,V_{x(B)})$. By the above construction, z is not contained in any element of $\gamma_m(x(B^*))$ for $B^*(\neq B)\in\mathscr{L}_n$. Thus we have $\mathrm{St}(z, \gamma_{n,m})=\mathrm{St}(z, \gamma_m(x(B)))$. Since $\mathrm{Int}_{(\mathrm{Cl}\,V_{x(B)})}\mathrm{St}(z, \gamma_m(x(B)))$ is open in $\mathrm{Cl}\ V_{x(B)}$, there is an open set G in X such that $G\cap \mathrm{Cl}\,V_{x(B)}=\mathrm{Int}_{(\mathrm{Cl}\,V_{x(B)})}\mathrm{St}(z,\ \gamma_m(x(B)))$. On the other hand $G\cap V'_{x(B)}$ is open in X, therefore we have $\mathrm{Int}\ \mathrm{St}(z, \gamma_m(x(B)))\supset G\cap V'_{x(B)}\ni z$. Hence we obtain (n, m) such that $z\in \mathrm{Int}\ \mathrm{St}(z, \gamma_{n,m})\subset \mathrm{St}(z, \gamma_{n,m})\subset U$.

Next we have $z\in \mathrm{Int}\ \mathrm{St}(z, \gamma_{k,l})$ for each $k, l\in N$. Because if $z\in V_{x(B)}$ for some $B\in \mathscr{L}_k$, then $z\in \mathrm{Int}_{(\mathrm{Cl}\,V_{x(B)})}\mathrm{St}(z, \gamma_l(x(B)))$. Thus we have $z\in \mathrm{Int}\,\mathrm{St}(z, \gamma_{k,l})$ by the above way. If $z\in V_{x(B)}$ for all $B\in\mathscr{L}_k$, then $z\in Q_k$. Since Q_k is open, therefore we have $z\in \mathrm{Int}\,\mathrm{St}(z, \gamma_{k,l})$.

Thus we have the following proposition: (1) γ is a semidevelopment.

By the similar way, we can prove the following proposition: (2) δ is a semidevelopment.

Next, let $\gamma'_{n,m}$ be an arbitrary subfamily of $\gamma_{n,m}$. If $Q_n\notin\gamma'_{n,m}$, then we put $\gamma_m(x(B))^*=\gamma_m(x(B))\cap\gamma'_{n,m}$. Since $\{\mathrm{Cl}\,V_{x(B)}\mid B\in\mathscr{L}_n\}$ is discrete and $\cup\{G\mid G\in\gamma_m(x(B))^*\}\subset \mathrm{Cl}\,V_{x(B)}$, we have

$$\begin{aligned}
\mathrm{Cl}(\cup\gamma'_{n,m}) &=\mathrm{Cl}(\cup\{G\mid G\in\gamma_m(x(B))^*,\ B\in\mathscr{L}_n\})\\
&=\mathrm{Cl}(\cup_{B\in\mathscr{L}_n}(\cup\{G\mid G\in\gamma_m(x(B))^*\}))\\
&=\cup_{B\in\mathscr{L}_n}(\mathrm{Cl}(\cup\{G\mid G\in\gamma_m(x(B))^*\}))\\
&=\cup_{B\in\mathscr{L}_n(\mathrm{Cl}\,V_{x(B)})}(\cup\{G\mid G\in\gamma_m(x(B))^*\}))\\
&\subset\cup_{B\in\mathscr{L}_n}(\cup\{D(G)\mid G\in\gamma_m(x(B))^*\})\\
&=\cup\{D(G)\mid G\in\gamma'_{n,m}\}.
\end{aligned}$$

If $Q_n\in\gamma'_{n,m}$,

$$\begin{aligned}
\mathrm{Cl}(\cap r'_{n,m}) &=\mathrm{Cl}(\cup\{G\mid G\in\gamma'_{n,m}, G\neq Q_n\})\cup \mathrm{Cl}\ Q_n\\
&\subset\cup\{D(G)\mid G\in\gamma'_{n,m}, G\neq Q_n\}\cup \mathrm{Cl}\ Q_n\\
&=\cup\{D(G)\mid G\in\gamma'_{n,m}\}.
\end{aligned}$$

Therefore, (3) $\gamma_{n.m}$ is cushioned in $\delta_{n.m}$. By (1), (2) and(3), the theorem is proved completely.

Burke [**5**] has shown that a locally developable space is developable if it is subparacompact. we can obtain analogous result as follows:

THEOREM 3. *A locally semidevelopable space is semidevelopable if and only if it is subparacompact.*

Proof. The necessity is proved by the present author [8] and the sufficiency can be proved by the similar way to theorem 4 and Burke's method.

Alexander has proved that a space is semi-metrizable if and only if it is a semi-developable T_0-space. It is well known that a locally T_0-space is T_0-space. Then by the Alexander's result and Theorem 3, we have the following

COROLLARY 4. *A locally semi-metric space is semi-metrizable if and only if it is subparacompact.*

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