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THE STRUCTURE OF $\mathfrak{L}(M, L)$

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1. Introduction

The global theory of the space $\mathfrak{L}(E, F)$ of all order bounded linear mappings from a linear lattice E into a linear lattice F is actively investigated. In particular, when F is an (L)-space, many theorems in the case F is the space of real numbers can be extended. In this connection we shall give a structure theorem of $\mathfrak{L}(M, L)$ (*cf.* Def. 1). We shall also give some remarks on the structure of $\mathfrak{L}(L, M)$ as a dual case.

For definitions we refer to Kelley and Namioka [**2**] and for elementary calculations we refer to Vulikh [**1**].

2. Definitions and notations.

Throughout this paper M is an abstract (M)-space with an order unit e and L is an abstract (L)-space. We say that a subset A of M (or L) is *order bounded* if A is contained in an interval

$$[x, y] = \{z \in M \text{ (or } L) \mid x \leqq z \leqq y\}.$$

DEFINITION 1. $\mathfrak{L}(M, L)$ (resp. $\mathfrak{L}(L, M)$) is the space of all the linear mappings from M(resp. L) into L(resp. M) which map every order bounded set in M (resp. L) to an order bounded set in L (resp. M).

3. Theorem

THEOREM 1. *$\mathfrak{L}(M, L)$ is an abstract (L)-space under the norm $\|\varphi\| = \|\sup_{\|x\| \leq 1} |\varphi(x)|\|$ for any $\varphi \in \mathfrak{L}(M, L)$.*

Proof. We notice that $\|\varphi\| = \|\sup_{\|x\| \leq 1} |\varphi(x)|\|$, which clearly exists. $\|\varphi\|$ is a norm for $\mathfrak{L}(M, L)$. In fact, if $\|\varphi\| = 0$, then $\sup_{\|x\| \leq 1} |\varphi(x)| = 0$ and hence $\varphi = 0$.

$$\begin{aligned} \|\alpha\varphi\| &= \|\sup_{\|x\| \leq 1} |\alpha\varphi(x)|\| \\ &= \| |\alpha| \sup_{\|x\| \leq 1} |\varphi(x)| \| \\ &= |\alpha| \| \sup_{\|x\| \leq 1} |\varphi(x)| \| \\ &= |\alpha| \|\varphi\| \end{aligned}$$

for any scalar α and any $\varphi \in \mathfrak{L}(M, L)$.

$$\begin{aligned} \|\varphi + \psi\| &= \|\sup_{\|x\| \leq 1} |(\varphi + \psi)(x)|\| \\ &= \|\sup_{\|x\| \leq 1} |\varphi(x) + \psi(x)|\| \\ &\leqq \|\sup_{\|x\| \leq 1} |\varphi(x)| + \sup_{\|x\| \leq 1} |\psi(x)|\| \\ &= \|\sup_{\|x\| \leq 1} |\varphi(x)|\| + \|\sup_{\|x\| \leq 1} |\psi(x)|\| \end{aligned}$$

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$$=\|\varphi\|+\|\psi\|$$

for any $\varphi, \psi\in\mathcal{L}(M, L)$.

The norm $\|\varphi\|$ is compatible with the order, that is, if $|\varphi|\leqq|\psi|$, then $\|\varphi\|\leqq\|\psi\|$. Clearly our norm is monotonic on the positive cone of $\mathcal{L}(M, L)$, that is, if $0\leqq\varphi\leqq\psi$, then $\|\varphi\|\leqq\|\psi\|$. Therefore it is enough to show that $\|\varphi\|=\||\varphi|\|$ for any $\varphi\in\mathcal{L}(M, L)$. But

$$\begin{aligned}\||\varphi|\| &= \|\sup_{\|x\|\leqq 1}||\varphi|(x)|\| \\ &= \|\sup_{\|x\|\leqq 1, x\geqq 0}||\varphi|(x)|\| \\ &= \|\sup_{\|x\|\leqq 1, x\geqq 0}|\varphi|(x)\| \\ &= \|\sup_{\|x\|\leqq 1, x\geqq 0}\sup_{|y|\leqq x}|\varphi(y)|\| \\ &= \|\sup_{\|x\|\leqq 1, x\geqq 0}|\varphi(x)|\| \\ &= \|\varphi\|.\end{aligned}$$

Now let us prove that our norm is additive on the positive cone of $\mathcal{L}(M, L)$. We notice that if $\varphi\in\mathcal{L}(M, L)$ and $\varphi\geqq 0$, then $\|\varphi\|=\|\varphi(e)\|$. Therefore, if $\varphi, \psi\in\mathcal{L}(M, L)$ and $\varphi\geqq 0, \psi\geqq 0$, then

$$\begin{aligned}\|\varphi+\psi\| &= \|(\varphi+\psi)(e)\| \\ &= \|\varphi(e)+\psi(e)\| \\ &= \|\varphi(e)\|+\|\psi(e)\|.\end{aligned}$$

Hence $\|\varphi+\psi\|=\|\varphi\|+\|\psi\|$.

To finish our proof, it remains to show that $\mathcal{L}(M, L)$ is a Banach space. To prove this, it is sufficient to prove that

1) if a sequence $\{\varphi_\alpha\}$ $(\varphi_\alpha\geqq 0)$ is decreasing and converges to zero in order, then $\{\varphi_\alpha\}$ converges to 0 in norm, and that

2) if a sequence $\{\varphi_\alpha\}$ $(\varphi_\alpha\geqq 0)$ is increasing without order bound, then sequence of norms $\{\|\varphi_\alpha\|\}$ increases without bound. (*cf.* Vulikh [**1**])

But $\|\varphi_\alpha\|=\|\varphi_\alpha(e)\|$ and $(\inf_\alpha\varphi_\alpha)(e)=\inf_\alpha(\varphi_\alpha(e))=0$. Hence 1) holds. If $\{\varphi_\alpha\}$ is increasing without bound, then so is $\varphi_\alpha(e)$ and hence $\|\varphi_\alpha(e)\|$ is increasing without bound.

This completes our proof.

We shall state some remarks on $\mathcal{L}(L, M)$.

REMARK 1. *If L has an order unit u and M is Dedekind complete, then $\mathcal{L}(L, M)$ is a normed lattice.*

Proof. We shall adopt the same norm as in the theorem 1, namely, $\|\varphi\|=\|\sup_{|x|\leqq u}|\varphi(x)|\|$ for any $\varphi\in\mathcal{L}(L, M)$. The same reasoning as in the first part of the proof of the theorem 1 concludes our assertion.

REMARK 2. *For arbitrary L and M with unit e*

$$\mathcal{L}(L, M)\supset\mathcal{L}_b(L, M)$$

where $\mathcal{L}_b(L, M)$ *is the space of all the norm bounded linear mappings from* L *into* M.

Proof. Let $\varphi \in \mathcal{L}_b(L, M)$. Any order bounded set is norm bounded in L. Hence φ maps an order bounded set to a norm bounded set in M which is order bounded.

$\mathcal{L}_b(L, M)$ is a Banach space under the usual supremum norm and carries a natural partial order.

REMARK 3. *The partially ordered Banach space* $\mathcal{L}_b(L, M)$ *carries an order unit. Moreover, our norm satisfies that*

$$\|\varphi \vee \psi\| = \|\varphi\| \vee \|\psi\|$$

for any positive elements φ, ψ *in* $\mathcal{L}_b(L, M)$.

Proof. Let e be the order unit of M. The mapping $u(x) = \|x\| e$ for positive element x in L is additive. Therefore it has a linear extension, say u again, on L. For x positive and $\varphi \in \mathcal{L}_b(L, M)$ it is true that

$$\varphi(x) \leqq \|\varphi(x)\| e \leqq \|\varphi\| \|x\| e = \|\varphi\| u(x)$$

and hence $\varphi \leqq \|\varphi\| u$. It follows that u is a unit and moreover, for positive elements φ and ψ of $\mathcal{L}_b(L, M)$, because of the inequality

$$\varphi \vee \psi \leqq (\|\varphi\| \vee \|\psi\|) u,$$

it is true that

$$\|\varphi \vee \psi\| \leqq (\|\varphi\| \vee \|\psi\|) \|u\| = \|\varphi\| \vee \|\psi\|.$$

This completes our proof.

References

[1] B. Vulikh; *Introduction to the theory of partially ordered spaces.* Wolters-Noordhoff Pub. Co., 1967.

[2] J. Kelley & I. Namioka; *Linear topological spaces.* Van Nostrand Pub. Co., 1963.

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