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ON LANDSBERG SPACES OF TWO DIMENSIONS WITH (α, β)-METRIC

Dedicated to Professor Dr. J. Kanitani on the occasion of his eightieth birthday

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In the theory of Finsler spaces of E. Cartan [**6**]*, there are three kinds of curvature tensors R_{hijk}, P_{hijk} and S_{hijk}. From the standpoint of the theory of connections in fibre bundles [**14**], the euclidean connection of Cartan is given by the pair (Γ^h, Γ^v) of distributions in a certain principal bundles which satisfies a system of axioms. Then the first R_{hijk}(resp. the third S_{hijk}) is thought of as the curvature tensor of Γ^h (resp. Γ^v), while the second P_{hijk}, the subject of the present paper, may be regarded as the mixed curvature tensor of Γ^h and Γ^v.

As to the first R_{hijk}, it was proved by H. Akbar-Zadeh [**1**] that if R_{hijk} is of the form $R_{hijk}=R(g_{hj}g_{ik}-g_{hk}g_{ij})$ and $R\neq 0$, then the third S_{hijk} vanishes. But we have not yet any essential result on the spaces with $R_{hijk}=0$, though a locally Minkowskian space is characterized by $R_{hijk}=0$ together with $C_{hij|k}=0$. As to the third S_{hijk}, it was proved by F. Brickell [**5**], [**19**] that if $S_{hijk}=0$ and the metric is symmetric, then the space is Riemannian.

The character of the space with $P_{hijk}=0$ has not yet been made clear in spite of efforts by several authors. The condition $P_{hijk}=0$ is equivalent to $C_{hij|0}=0$ [**3**], [**4**]. Such a space of two dimensions was first considered by G. Landsberg [**12**], so that we shall call such a space of general dimension as a *Landsberg space* following A. Moór [**16**]. A Finsler space satisfying a stronger condition $C_{hij|k}=0$ is called to be affinely connected by L. Berwald, who found all of such spaces of two dimensions [**2**].

The purpose of the present paper is to contribute a little to the theory of Landsberg spaces of two dimensions. The main results are stated in Theorems 1 and 3. It is an interesting conclusion that some complex analytic functions are closely related to these spaces. Does this situation happen through just such a special metric, or does it arise intrinsically from the condition $P_{hijk}=0$?

We wish to express our sincere thanks to Professor Dr. Y. Ohya for his valuable advice to solve a system of partial differential equations appearing in §4.

§1. Finsler spaces with (α, β)-metric

M. Matsumoto, one of the authors, has recently treated Finsler spaces with (α, β)-metric [**15**], which is a special simple metric as follows:

DEFINITION. *A Finsler space is called to be with an (α, β)-metric when the fundamental function $L=L(\alpha, \beta)$ is a positively homogeneous function of degree one in*

$$\alpha(x, y)=(a_{ij}(x)y^iy^j)^{1/2} \text{ and } \beta(x, y)=b_i(x)y^i,$$

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* Numbers in brackets refer to the references at the end of the paper.

where $ds^2=\alpha^2(x, dx)$ *is a Riemannian metric,* b_i *is a non-zero covariant vector field and* $y^i=\dot{x}^i$.

This metric is a generalization of the relativistic Finsler metric $L=\alpha+\beta$ adopted to make the indicatrix eccentric, which was introduced by G. Randers [18] and has been studied by many authors. (See, for instance, [7], [8], [9], [13], [17], [20].) The metric $L=\alpha^2/\beta$ treated by V. K. Kropina [10], [11] belongs also to this class of metric.

The following results are known as to Finsler spaces with (α, β)-metric. First, the fundamental function $L(\alpha, \beta)$ of any C-reducible Finsler space with (α, β)-metric is essentially either the Randers metric $\alpha+\beta$ or the Kropina metric α^2/β. Here a Finsler space of dimension more than two is called C-reducible if it is not Riemannian and the torsion tensor C_{ijk} is of the form $C_{ijk}=(h_{ij}C_k+h_{jk}C_i+h_{ki}C_j)/(n+1)$, where $h_{ij}=g_{ij}-l_il_j$ and $C_i=C_{ijk}g^{jk}$. Secondly, a C-reducible Landsberg space is an affinely connected space of Berwald ($C_{hij|k}=0$). It should be emphasized that these results are proved under the condition that the dimension is more than two.

In this paper we shall restrict ourselves to consider Finsler spaces of two dimensions with (α, β)-metric. Then, throughout the paper, we shall refer to an *isothermal coordinate* (x^i), with respect to which the α is written in the form

$$\alpha=a(x)\mu, \quad \text{where } \mu=((y^1)^2+(y^2)^2)^{1/2}. \tag{1.1}$$

Let us denote by z^i the vector orthogonal to the supporting element y^i and having the same length with y^i with respect to the metric α:

$$z^i=\varepsilon_{ij}y^j, \text{ i.e., } z^1=y^2, \ z^2=-y^1. \tag{1.2}$$

If we put

$$\gamma=b_iz^i, \tag{1.3}$$

it is easily obtained that

$$\frac{\partial\alpha}{\partial y^i}=\frac{1}{\mu^2}\alpha y^i, \quad \frac{\partial\beta}{\partial y^i}=\frac{1}{\mu^2}(\beta y^i+\gamma z^i)=b_i, \quad \frac{\partial\gamma}{\partial y^i}=\frac{1}{\mu^2}(\gamma y^i-\beta z^i). \tag{1.4}$$

The following lemma will be easily shown by Euler theorem on homogeneous functions and (1.4).

LEMMA 1. *Let* $f(\alpha, \beta, \gamma, x^i)$ *be a positively homogeneous function of degree* p *in* α, β and γ. *Then*

$$\frac{\partial f}{\partial y^i}=\frac{1}{\mu^2}(pfy^i+f_{(y)}z^i),$$

where the y-operation $f \to f_{(y)}$ *is defined by*

$$f_{(y)}=\gamma f_\beta-\beta f_\gamma=\frac{1}{\gamma}(\alpha\beta f_\alpha+(\beta^2+\gamma^2)f_\beta-p\beta f).$$

Throughout the paper, the subscripts α, β, γ will be used to denote partial differentiations by α, β, γ respectively. It is remarked that $f_{(y)}$ is also a positively homogeneous function of degree p in α, β and γ.

Next we put

$$A=\frac{\partial\alpha}{\partial x^j}y^j=\mu a_{,j}y^j,\quad B=\frac{\partial\beta}{\partial x^j}y^j=b_{i,j}y^iy^j, \tag{1.5}$$
$$C=\frac{\partial\gamma}{\partial x^j}y^j=b_{i,j}z^iy^j,$$

where $(,_j)$ will be used to denote the differentiation by x^j.

LEMMA 2. *Let $f(\alpha, \beta, \gamma)$ be a positively homogenous function of degree p in α, β and γ. Then*

$$\frac{\partial f}{\partial x^j}y^j=\frac{1}{\gamma}(pCf+f_{(x)}),$$

where the x-operation $f \rightarrow f_{(x)}$ is defined by

$$f_{(x)}=(A\gamma-C\alpha)f_\alpha+(B\gamma-C\beta)f_\beta.$$

It is remarked that $f_{(x)}$ is also a positively homogeneous function of degree $p+2$ in α, β and γ.

§2. The main scalar I and its derivative I_s

In the theory of two-dimensional Finsler spaces, the main scalar I plays an important role, which was introduced by L. Berwald [**2**] and is defined by $LC_{ijk}=Im_im_jm_k$, where m_i is the unit vector orthogonal to the supporting element y^i. The main scalar I is also written in the from

$$I=\frac{D}{2(LF^3)^{1/2}}, \tag{2.1}$$

where we put

$$F=\frac{1}{(y^2)^2}\frac{\partial^2 L}{\partial y^1\partial y^1}=\frac{g}{L^3},\qquad g=\det.(g_{ij}), \tag{2.2}$$

and

$$D=\frac{\partial L}{\partial y^1}\frac{\partial F}{\partial y^2}-\frac{\partial L}{\partial y^2}\frac{\partial F}{\partial y^1}.$$

In virtue of Lemma 1, we obtain

$$F=\frac{E}{\mu^4},\ \text{where}\qquad E=\alpha L_\alpha+\gamma^2 L_{\beta\beta}, \tag{2.3}$$

and

$$D=-\frac{1}{\mu^6}(3EL_{(y)}+LE_{(y)}).$$

Consequently we obtain

PROPOSITION 1. *The main scalar I of a two-dimensional Finsler space of (α,β)-metric $L(\alpha,\beta)$ is given by*

$$I=-\frac{3EL_{(y)}+LE_{(y)}}{2(LE^3)^{1/2}},$$

which is the positively homogeneous function of degree zero in α,β and γ.

We now apply the above to special cases which will be treated in detail. Let us first consider the Randers metric $L=\alpha+\beta$. Then it is easy to show

PROPOSITION 2. *The main scalar I of a two-dimensional Finsler space with the Randers metric $L=\alpha+\beta$ is given by*

$$I=-\frac{3\gamma}{2((\alpha+\beta)\alpha)^{1/2}}.$$

Let us introduce a direct generalization of the Kropina metric $L=\alpha^2/\beta$.

DEFINITION. *A special (α,β)-metric $L=\alpha^{m+1}/\beta^m, m\neq 0, -1,$ is called a generalized Kropina metric.*

As to this metric, if we regard α as μ and $a^{-(m+1)/m}b_i$ as b_i, then L is written in the form

$$L=\frac{\mu^{m+1}}{\beta^m}, \quad (\alpha=\mu,\ m\neq 0,\ -1). \tag{2.4}$$

It is easy to show

PROPOSITION 3. *The main scalar I of a two-dimensional Finsler space of a generalized Kropina metric $L=\mu^{m+1}/\beta^m$ is written in the form*

$$I=\frac{m(m+1)\gamma\{3\beta^2+(2m+1)\gamma^2)\}}{\{(m+1)(\beta^2+m\gamma^2)\}^{3/2}}.$$

It is remarked that this I is a positively homogeneous function of degree zero in β and γ, independent of α $(=\mu)$.

We are now concerned with the derivative I_s of the main scalar I:

$$LI_s=\left(\frac{\partial I}{\partial x^j}-\frac{\partial I}{\partial y^i}N^i{}_j\right)y^j,$$

where $N^i{}_j$ are coefficients of the non-linear connection subordinating to the Cartan connection. If we denote by $\gamma_k{}^i{}_j$ the Christoffel symbols constructed from the fundamental tensor $g_{ij}(x,y)$ with respect to x^i, it is well-known that $N^i{}_jy^j=\gamma_k{}^i{}_hy^ky^h$, so that LI_s is written in the form

$$LI_s=\frac{\partial I}{\partial x^j}y^j+L\frac{\partial I}{\partial y^i}\left(\frac{\partial L}{\partial x^j}-\frac{\partial^2 L}{\partial x^k\partial y^j}y^k\right)g^{ij}. \tag{2.5}$$

Let us find a more convenient expression of I_s in the case of (α,β)-metric. First, by virtue of Lemmas 1 and 2, we obtain

$$\frac{\partial I}{\partial x^j} y^j = \frac{1}{\gamma} I_{(x)}, \qquad I_{(x)} = (A\gamma - C\alpha) I_\alpha + (B\gamma - C\beta) I^\beta,$$

$$\frac{\partial I}{\partial y^i} = \frac{1}{\mu^2} I_{(y)} z^i, \quad I_{(y)} = \frac{1}{\gamma} (\alpha\beta I_\alpha + (\beta^2 + \gamma^2) I_\beta).$$

Secondly, if we put

$$A^* = \mu a_{,j} z^j, \tag{2.6}$$

then the first of (1.5) and (2.6) give $a_{,j} = (Ay^j + A^* z^j)/\mu^3$, hence

$$\frac{\partial L}{\partial x^j} = \frac{L_\alpha}{\mu^2} (Ay^j + A^* z^j) + L_\beta b_{k,j} y^k.$$

On the other hand, we see

$$y^k \frac{\partial}{\partial y^j} \left(\frac{\partial L}{\partial x^k} \right) = y^k \frac{\partial}{\partial y^j} (L_\alpha \alpha_{,k} + L_\beta \beta_{,k})$$

$$= y^k \left(\frac{1}{\mu^2} L_{\alpha(y)} z^j \alpha_{,k} + L_\alpha a_{,k} \frac{y^j}{\mu} + \frac{1}{\mu^2} L_{\beta(y)} z^j \beta_{,k} + L_\beta b_{j,k} \right).$$

It then follows from (1.5) that

$$\frac{\partial L}{\partial x^j} - \frac{\partial^2 L}{\partial x^k \partial y^j} y^k = \frac{H}{\mu^2} z^j, \tag{2.7}$$

where we put

$$H = A^* L_\alpha - \mu^2 b_{12} L_\beta + \frac{\gamma}{\alpha} (A\beta - B\alpha) L_{\beta\beta}, \tag{2.8}$$

and $b_{12} = b_{1.2} - b_{2.1}$. Paying attention to (2.2), (2.3) and $g^{ij} z^i z^j = L^2/g$, we obtain finally

PROPOSITION 4. *The derivative I_s of the main scalar I of a two-dimensional Finsler space with (α, β)-metric $L(\alpha, \beta)$ is given by*

$$LI_s = \frac{1}{\gamma} \left(A\gamma - C\alpha + \frac{H}{E} \alpha\beta \right) I_\alpha + \frac{1}{\gamma} \left(B\gamma - C\beta + \frac{H}{E} b^2 \mu^2 \right) I_\beta,$$

where we put $b^2 = (b_1)^2 + (b_2)^2$.

We consider the particular case where the main scalar I does not contain α and $a=1$ $(\alpha = \mu)$ like in Proposition 3. In this case $A = A^* = 0$ from (1.5) and (2.6). Therefore we obtain

PROPOSITION 5. *In a special case where the main scalar I does not contain α and $a=1$, the derivative I_s of I is of the form*

$$L\, I_s = \frac{1}{E\gamma} ((B\gamma - C\beta) E - (\mu^2 b_{12} L_\beta + \gamma B L_{\beta\beta}) b^2 \mu^2) I_\beta.$$

§ 3. The Landsberg spaces with the Randers metric

We shall consider a two-dimensional Landsberg space with the Randers metric $L = \alpha + \beta$, i.e., $I_s = 0$. In this case we obtain $H = A^* - \mu^2 b_{12}$ from (2.8) and $E = \alpha = a\mu$

from (2.3). Putting $A=\mu A_0$ and $A^*=\mu A_0^*$, it follows from Proposition 4 that $I_s=0$ is equivalent to the equation

$$X_2\mu+X_3=0, \tag{3.1}$$

where we put

$$X_2=2a(A_0\gamma-Ca+A_0^*\beta)-(2\beta^2+\gamma^2)b_{12},$$
$$X_3=(A_0\gamma-Ca+A_0^*\beta)\beta+a(B\gamma-C\beta)+\mu^2(A_0^*b^2-2a\beta b_{12}). \tag{3.2}$$

In the equation (3.1), μ is given by (1.1), and X_2 (resp. X_3) is a polynomial of degree two (resp. three) in y^i. This observation enables us to state that (3.1) is equivalent to $X_2=X_3=0$. Thus, equating the coefficients of $(y^1)^2$ and $(y^2)^2$ in X_2 to zero, we get immediately

$$b_{1\cdot 2}=b_{2\cdot 1}=\frac{a_{,1}b_2+a_{,2}b_1}{a}. \tag{3.3}$$

Next, equating the coefficient of y^1y^2 in X_2 to zero, we obtain

$$b_{1\cdot 1}-b_{2\cdot 2}=\frac{2(a_{,1}b_1-a_{,2}b_2)}{a}. \tag{3.4}$$

In virtue of (3.3) and (3.4), we observe $X_3=a(B\gamma-C\beta)+A_0^*\mu^2b^2$. Accordingly, equating the coefficients of $(y^1)^3$ and $(y^2)^3$ in X_3 to zero, we get immediately

$$b_{1\cdot 1}=-b_{2\cdot 2}=\frac{a_{,1}b_1-a_{,2}b_2}{a}, \tag{3.5}$$

and it is easy to see that (3.4) and $X_3=0$ are satisfied by (3.3) and (3.5). Therefore the condition $I_s=0$ reduces to (3.3) and (3.5).

Finally (3.3) and (3.5) lead us to the following two facts: There exists a function $f(x)$ such that

$$b_i=\frac{\partial f}{\partial x^i},\quad \frac{\partial^2 f}{\partial x^1\partial x^1}+\frac{\partial^2 f}{\partial x^2\partial x^2}=0, \tag{3.6}$$

and $a=bk$ for a constant k. Therefore we establish

THEOREM 1. *A two-dimensional Finsler space with the Randers metric $L=\alpha+\beta$ is a Landsberg space if and only if the fundamental function L is written in the form*

$$L=k((b_1)^2+(b_2)^2)^{1/2}((y^1)^2+(y^2)^2)^{1/2}+(b^1y^1+b_2y^2),$$

where k is a constant and b_i is a gradient vector of a harmonic function of the variables x^i.

On the other hand, (3.3) and (3.5) show that $b_2(x)+ib_1(x), i^2=-1$, is an analytic function of the variable x^1+ix^2. Therefore we obtain an alternative characterization of the spaces under consideration:

THEOREM 1′. *A necessary and sufficient condition for a two-dimensional Finsler space of the Randers metric to be a Landsberg space is that $a/((b_1)^2+(b_2)^2)^{1/2}$ is a constant and b_2+ib_1 is a complex analytic function of the variable x^1+ix^2.*

§ 4. Landsberg spaces with a generalized Kropina metric

We shall consider a two-dimensional Landsberg space with a generalized Kropina metric $L=\alpha^{m+1}/\beta^m$, $m\neq 0$, -1. In this case we have already shown Proposition 5, so that the equation characterizing such a space is now written down in the form

$$(m+1)(\beta^2+m\gamma^2)((b_1b_{2.1}-b_2b_{1.1})y^1+(b_1b_{2.2}-b_2b_{1.2})y^2)+mb^2b_{12}\beta\mu^2-m(m+1)b^2B\gamma=0.$$

The left-hand side of the above is a polynomial of degree three in y^i. If we denote by X_{12}, Y_{12}, $-Y_{21}$ and $-X_{21}$ its coefficients of $(y^1)^3$, $(y^1)^2y^2$, $y^1(y^2)^2$ and $(y^2)^3$ respectively, then the above is equivalent to

$$X_{12}=b_1((m^2-1)B_{12}b_{1.1}+mb^2b_{1.2}+(B_{11}+m^2B_{22})b_{2.1})=0, \tag{4.1}$$

$$\begin{aligned}Y_{12}=&(m+1)b_1((-mB_{11}+(m-2)B_{22})b_{1.1}+(B_{11}+mB_{22})b_{2.2})\\&+b_2(((m^2+m-1)B_{11}+mB_{22})b_{1.2}+(-(m^2-2)B_{11}+m^2B_{22})b_{2.1})=0,\end{aligned} \tag{4.2}$$

and $X_{21}=Y_{21}=0$, where we put $B_{ij}=b_ib_j$ and X_{21} (resp. Y_{21}) is obtained from X_{12} (resp. Y_{12}) by interchange of subscripts 1 and 2. Moreover it is easy to show that

$$\begin{aligned}&B_{12}\{(m-2)B_{11}-mB_{22}\}Y_{12}-B_{11}(B_{11}+mB_{22})Y_{21}\\&=\{m(B_{11}{}^2-B_{11}B_{22}+B_{22}{}^2)+3B_{11}B_{22}\}X_{12},\end{aligned}$$

and the similar relation between Y_{12}, Y_{21} and X_{21}. These facts lead us to the result that a necessary and sufficient condition for a two-dimensional Finsler space with a generalized Kropina metric to be a Landsberg space is that b_i satisfy the differential equations $Y_{12}=Y_{21}=0$, assuming

$$m\{(b_1)^4-(b_1b_2)^2+(b_2)^4\}+3(b_1b_2)^2\neq 0. \tag{4.3}$$

In this stage, we shall treat the Kropina metric $(m=1)$. Then $Y_{12}=Y_{21}=0$ reduce easily to $b_{1.1}=b_{2.2}$ and $b_{1.2}=-b_{2.1}$, and the assumption (4.3) is $b^2\neq 0$ only. As a consequence we conclude

THEOREM 2. *A two-dimensional Finsler space with the Kropina metric $L=\mu^2/\beta$ is a Landsberg space if and only if b_1+ib_2 is a complex analytic function of the variable x^1+ix^2.*

It is interesting to see that b_1+ib_2 is analytic in Theorem 2, while b_2+ib_1 is analytic in Theorem 1′.

Now we shall turn to a consideration of the general case. The system of partial differential equations $Y_{12}=Y_{21}=0$ can be written in the matrix form

$$M\frac{\partial B}{\partial x^1}+N\frac{\partial B}{\partial x^2}=0, \tag{4.4}$$

where we put

$$B=\begin{pmatrix}b_1\\b_2\end{pmatrix},\quad M=\begin{pmatrix}M_{11}&M_{12}\\M_{21}&M_{22}\end{pmatrix},\quad N=\begin{pmatrix}N_{11}&N_{12}\\N_{21}&N_{22}\end{pmatrix},$$

$$M_{11}=(m+1)b_1(-mB_{11}+(m-2)B_{22}),\qquad M_{12}=b_2(-(m^2-2)B_{11}+m^2B_{22}),$$

$$M_{21}=(m+1)b_2(B_{22}+mB_{11}), \qquad M_{22}=b_1((m^2+m-1)B_{22}+mB_{11}),$$
$$N_{11}=b_2((m^2+m-1)B_{11}+mB_{22}), \qquad N_{12}=(m+1)b_1(B_{11}+mB_{22}),$$
$$N_{21}=b_1(-(m^2-2)B_{22}+m^2B_{11}), \qquad N_{22}=(m+1)b_2(-mB_{22}+(m-2)B_{11}).$$

The det. (M_{ij}) is equal to

$$-m(m+1)b^2\{m(b_1{}^4-(b_1b_2)^2+b_2{}^4)+3(b_1b_2)^2\},$$

which was assumed not to vanish in (4.3), so that we have the inverse matrix M^{-1} of M. Hence (4.4) is rewritten in the form

$$\frac{\partial B}{\partial x^1}+M^{-1}N\frac{\partial B}{\partial x^2}=0, \tag{4.4'}$$

where the matrix $M^{-1}N$ is given by

$$M^{-1}N=-\frac{1}{mb^2}\begin{pmatrix}(m^2-1)B_{12} & B_{11}+m^2B_{22}\\ -B_{22}-m^2B_{11} & -(m^2-1)B_{12}\end{pmatrix}.$$

This matrix has the characteristic roots i and $-i$, hence it is diagonalized by a matrix U:

$$U^{-1}(M^{-1}N)U=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}.$$

Such a matrix U is taken as

$$U=\begin{pmatrix}b_1+imb_2 & b_1-imb_2\\ b_2-imb_1 & b_2+imb_1\end{pmatrix}.$$

Because of the equality $2U^{-1}B=\begin{pmatrix}1\\1\end{pmatrix}$, it is seen that (4.4′) is equivalent to

$$\left(U^{-1}\frac{\partial U}{\partial x^1}+\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}U^{-1}\frac{\partial U}{\partial x^2}\right)\begin{pmatrix}1\\1\end{pmatrix}=0. \tag{4.5}$$

Then, equating the real and imaginary parts of (4.5) to zero, we obtain easily

$$\frac{m}{2}\frac{\partial}{\partial x^1}(\log b^2)=\frac{\partial}{\partial x^2}\left(\tan^{-1}\frac{b_2}{b_1}\right),$$
$$\frac{m}{2}\frac{\partial}{\partial x^2}(\log b^2)=-\frac{\partial}{\partial x^1}\left(\tan^{-1}\frac{b_2}{b_1}\right).$$

These are the equations of Cauchy-Riemann type, so that we conclude

THEOREM 3. *A two-dimensional Finsler space with a generalized Kropina metric* $L=\mu^{m+1}/\beta^m, m\neq 0, -1$, *is a Landsberg space if and only if*

$$m\log((b_1)^2+(b_2)^2)^{1/2}+i\tan^{-1}\frac{b_2}{b_1}$$

is a complex analytic function of the variable x^1+ix^2, *assuming* (4.3).

If we put

$$u(x^1, x^2)+iv(x^1, x^2)=m\log((b_1)^2+(b_2)^2)^{1/2}+i\tan^{-1}\frac{b_2}{b_1},$$

then we obtain $b_1=e^{u/m}\cos v$ and $b_2=e^{u/m}\sin v$, hence the generalized Kropina metric L of the Landsberg space is written down as

$$L=\frac{((y^1)^2+(y^2)^2)^{(m+1)/2}}{e^u(y^1\cos v+y^2\sin v)^m}, \tag{4.6}$$

where u (resp. v) is a real (resp. imaginary) part of an analytic function of the variable x^1+ix^2. It is seen from the theory of surfaces that (u, v) is generally an isothermal coordinate with respect to the Riemannian metric α, because (x^1, x^2) has been taken as an isothermal coordinate from the first.

It is finally remarked that Theorems 2 and 3 were obtained under the assumption (4.3). Since our attention has been confined to the real b_1 and b_2, the assumption does not hold only in the case $-3\leqq m<1$.

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