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ON EVALUATION OF PREORDERS

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1. Introduction

In most cases of matrix game situtions, utilities are given in real numbers. But it is also possible to be given in members of a preordered set such as reflexive-transitive preference order. In such cases, it is sometimes necessary to evaluate these preorders into complete orders so that a decision maker is able to apply decision criteria to choose an optimal strategy.

In Chapter 2, we shall discuss some concepts related to preorders. Most of this chapter consists of well-known facts, but new concepts will also be introduced, e. g. a preordered set with $(1,\alpha)$-span. Furthermore, we shall make use of the diagrammatic representation of preorders which illustrates the structure of rather complicated preordered sets.

In Chapter 3, we shall present four ways to derive evaluations from a preordered set, and exhibit conditions under which evaluations are equivalent.

2. Preliminaries

Let P be a nonempty set. A binary relation on P is called a *preorder* if it is reflexive and transitive, and will be usually denoted by $\gtrsim$, or by $\lesssim$, or by similar types of symbols, in this paper. The pair $(P, \gtrsim)$, or the pair $(P, \lesssim)$, is called a *preordered set*, or shorter, a *preoset*.

If the symbols $\gtrsim$ and $\lesssim$ are used in the same context for binary relations on P, then each is assumed to designate the *inverse relation* of the other, i. e., for every $x, y\in P$,

$$x\gtrsim y \Longleftrightarrow y\lesssim x;$$

if one of them is a preorder, then obviously the other one is also a preorder.[1)]

The binary relation on P defined as the negation of the union of the preorder $\gtrsim$ and its inverse preorder $\lesssim$ will be denoted by $/\!\!/$ and called the *incomparability relation* derived from $\gtrsim$; it is obviously symmetric, but not reflexive and not transitive.

The binary relation on P defined as the intersection of the preorder $\gtrsim$ and its inverse preorder $\lesssim$ is obviously an equivalence relation and called the *indifference relation of P induced by* $\gtrsim$ and will be denoted by $\sim$ in this paper. Thus

$$x\sim y \Longleftrightarrow x\gtrsim y \text{ and } x\lesssim y.$$

The negation of $\sim$ will be denoted by $\nsim$. The binary relation on P defined as the intersection of a preorder $\gtrsim$, or $\lesssim$, and the negation of its induced indifference relation will be denoted by $>$, or $<$, respectively. Thus

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1) This is the *duality principle* of Birkhoff [2], extended from posets to preosets.

$$x > y \Leftrightarrow x \gtrsim y \text{ and } x \not\sim y.$$

In particular, $x > y$ implies $x \neq y$.

A mapping $\theta: (P, \gtrsim) \rightarrow (P^*, *\gtrsim)$ is called *strongly isotone* if it satisfies

$$x \sim y \Longrightarrow \theta(x) \sim^* \theta(y) \text{ and } x > y \Longrightarrow \theta(x) {}^*\!\!> \theta(y)$$

for every $x, y \in P$.

A mapping θ; $(P, \gtrsim) \rightarrow (P^*, *\gtrsim)$ is called *strongly antitone* if it satisfies

$$x \sim y \Longrightarrow \theta(x) \sim^* \theta(y), \text{ and } x > y \Longrightarrow \theta(x) <^* \theta(y)$$

for every $x, y \in P$.

A bijection θ: $(P, \gtrsim) \rightarrow (P^*, *\gtrsim)$ is called an *isomorphism* or a *preorder-isomorphism* if it is strongly isotone in both directions ; $(P, \gtrsim)$ and $(P^*, *\gtrsim)$ are said to be *isomorphic* or *preorder-isomorphic*, and we shall write

$$(P, \gtrsim) \cong (P^*, *\gtrsim).$$

A bijection θ: $(P, \gtrsim) \rightarrow (P^*, *\gtrsim)$ is called a *dual-isomorphism* if it is strongly antitone in both directions; $(P, \gtrsim)$ and $(P^*, *\gtrsim)$ are said to be *dual-isomorphic.*

In particular, $(P, \gtrsim)$ and $(P^*, *\gtrsim)$ are said to be *equal*:

$$(P, \gtrsim) = (P^*, *\gtrsim)$$

if $P = P^*$, and $\gtrsim$ and $*\gtrsim$ are the same relations.

Various orders can be derived from a preorder, that is; a preorder $\gtrsim$ on P is called a *partial order* if the following holds:

$$x \gtrsim y \text{ and } x \lesssim y \Longrightarrow x = y$$

for every $x, y \in P$, and $(P, \gtrsim)$ is called a *partially ordered set*, or, shorter, a poset.

A preorder $\gtrsim$ on P is called a *rank-order*[1] if, for every $x, y \in P$,

$$x \gtrsim y \text{ or } x \lesssim y$$

holds, and $(P, \gtrsim)$ is called a *rank-ordered set.*

A partial order $\gtrsim$ on P which is at the same time a rank order is called a *complete order*, and $(P, \gtrsim)$ is called a *completely ordered set.*

In a preoset $(P, \gtrsim)$, we shall define the *distance* between two elements x and y, to be denoted both $d(x, y)$ or by $d(y, x)$, as follows:

$d(x, y) = \infty$, if $x \not\parallel y$,

$= -1 +$ the number of equivalence classes of the shortest chain from $[x]$[2] to $[y]$ or from $[y]$ to $[x]$, if $x \gtrsim y$ or $y \gtrsim x$.

The distance between a maximal and a minimal element of a finite chain will be called the *length* of the chain.

1) For a detailed discussion of rank-orders, see, e.g., [12], and also [7].

2) [ ] denotes the equivalence class by the relation $\sim$.

A preoset $(P, \gtrsim)$ is said to be *decomposable* if there exist two non-empty subpreosets $(P_1, {}_1\gtrsim)$ and $(P_2, {}_2\gtrsim)$ of it such that the following holds;

$$P_1 \cap P_2 = \phi, \quad P_1 \cup P_2 = P \text{ and } x \not\!/ y$$

for every $x \in P_1, y \in P_2$.

A preoset which is not decomposable is said to be *connected.*

A connected preoset is said to have an $(1, \alpha)$-span if there exists a positive integer α such that every element belongs to a maximal chain of length α and any maximal chain which does not have length α must have length 1.

A *direct sum* of two disjoint preosets $(P_1, {}_1\gtrsim)$ and $(P_2, {}_2\gtrsim)$ is defined by

$$(P_1, {}_1\gtrsim) \dot{+} (P_2, {}_2\gtrsim) = (P_1 \cup P_2, \gtrsim)$$

where $\gtrsim = {}_1\gtrsim$ on P_1 and $\gtrsim = {}_2\gtrsim$ on P_2, and, for every $x \in P_1$ and $y \in P_2$, $x \not\!/ y$ with respect to the incomparability relation derived from $\gtrsim$, and $(P_1, {}_1\gtrsim)$ and $(P_2, {}_2\gtrsim)$ are called *direct summands* of $(P_1 \cup P_2, \gtrsim)$.

Let P be a finite set with a given preorder $\gtrsim$. Let $R (\in P \times P)$ be a minimal binary relation on P such that this preorder is the reflexive-transitive closure of R. [1)]

Then, the preorder can be represented by an *arrow diagram* in the following way:

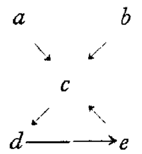
First, to each pair $(x, y) \in R$ we assign an arrow leading from the point (assigned to) x to the point (assigned to) y.

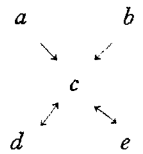
Now for any two elements $x, y \subset P$ we have

$$x \gtrsim y$$

if and only if either $x = y$ or there exists a sequence of arrows leading from the point x to the point y, possibly via intermediate points, in the usual way. (In the language of graph theory (P, R) is called a *basis graph* [**10**] and R a *line basis* [**6**] of the graph $(P, \gtrsim)$, and the preorder $\gtrsim$ is the *reachability relation* derived form R [**6**].)

EXAMPLE 1. Let $Q = \{a, b, c, d, e\}$ and $(Q, \gtrsim)$ satisfy $a > c$, $b > c$, and $c \gtrsim d \gtrsim e \gtrsim c$. Then, $(Q, \gtrsim)$ can be represented by either of the following arrow-diagrams;





3. Evaluations of preorders

Given a preordered set $(P, \gtrsim)$, a mapping from $(P, \gtrsim)$ into integers will be called an *evalvation.* Here we shall list four ways to evaluate a preordered set. Further we shall discuss relations between these ways.

In this chapter, the set of integers will be denoted by $\mathbf{N}$.

UE 1. The *first regret evaluation* $u_1 : (P, \gtrsim) \to (\mathbf{N}, \geqq)$ is defined as follows: For every

1) See [**7**].

$x \in P$,

$$u_1(x)=0, \text{ if } x \text{ is in the set of maximal elements of } (P, \gtrsim),$$
$$=-1, \text{ if } x \text{ is in the set of maximal elements of } (P \setminus \{y \mid u_1(y)=0, y \in P\}, \gtrsim),$$
$$=-t, \text{ if } x \text{ is in the set of maximal elements of}$$
$$(P \setminus \{y \mid u_1(y)=0, \ -1, \cdots\cdots, \ -(t-1), \ y \in P\}, \gtrsim).$$

In other words, $u_1(x) = -\max\{\text{finite distances from maximal elements of } (P, \gtrsim) \text{ to } x\}$.

UE 2. The *second regret evaluation* $u_2: (P, \gtrsim) \rightarrow (\mathbf{N}, \geqq)$ is defined as follows: For every $x \in P$,

$$u_2(x) = -\text{the number of all } y \text{ in } (P, \gtrsim) \text{ such that } y > x.$$

EXAMPLE 2: Let $(Q, \gtrsim)$ be as in Example 1 of Chapter 2. Then, $u_1(c)=-1$, but $u_2(c)=-2$. Note, however, that for every $y \in Q$,

$$u_1(y)=0 \iff u_2(y)=0.$$

UE 3. The *first complacency evaluation* u_3: $(P, \gtrsim) \rightarrow (\mathbf{N}, \geqq)$ is defined as follows: For every $x \in P$,

$$u_3(x)=0, \text{ if } x \text{ is in the set of minmal elements of } (P, \gtrsim)$$
$$=1, \text{ if } x \text{ is in the set of minimal elements of}$$
$$(P \setminus \{y \mid u_3(y)=0, \ y \in P\}, \ \gtrsim),$$
$$=t, \text{ if } x \text{ is in the set of minimal elements of}$$
$$(P \setminus \{y \mid u_3(y)=0, 1, \cdots\cdots, \ (t-1), y \in P\}, \gtrsim).$$

In other words, $u_3(x) = \max$ {finite lengths of maximal chains from x to minimal elements of $(P, \gtrsim)$}.

UE 4. The *second complacency evaluation* $u_4: (P, \gtrsim) \rightarrow (\mathbf{N}, \geqq)$ is defined as follows: For every $x \in P$,

$$u_4(x) = \text{the number of all } y \text{ in } P \text{ such that } y < x.$$

EXAMPLE 3. Let $(Q, \gtrsim)$ be as in the example of the Chapter 2. Then $u_3(a)=1$, but $u_4(a)=3$. Note, however, that, for every $y \in Q$,

$$u_3(y)=0 \iff u_4(y)=0.$$

PROPOSITION 2. 1. *Let $(P, \gtrsim)$ and $(P^*, \lesssim^*)$ be two dual-isomorphic preosets. Let θ be the given dual-isomorphism from $(P, \gtrsim)$ onto $(P^*, \lesssim^*)$. Let $u_n (n=1, 2)$ be the n^{th} regret evaluation on $(P, \gtrsim)$, $u_m^* (m=3, 4)$ be the $(m-2)^{th}$ complacency evaluation on $(P^*, \lesssim^*)$, and ν be the mapping such that, for any integer n, $\nu(n)$ is $-n$. Then*

(1) $$u_n = \nu \circ u_m^* \circ \theta$$

and

(2) $$u_m^* = \nu \circ u_m \circ \theta^{-1}$$

hold,. i. e., *the following diagram is commutative:*

$$\begin{array}{ccc} P & \longrightarrow & \{u_n(x) \mid x \in P\} \\ \theta^{-1}\uparrow\downarrow\theta & & \nu\uparrow\downarrow\nu \\ P^* & \longrightarrow & \{u_m^*(x^*) \mid x^* \in P^*\} \end{array}$$

REMARK: From the proposition 2. 1, it follows that order reversal of the set followed first by the regret evaluation and then by the change of sign gives the same result as the complacency evaluation, and vice versa. In this sense, these two evaluations are dual.

Proof. It suffices to prove (1), because (2) is an immediate consequence of (1).

First we considerthe case $n=1$ and $m=3$. By the definitions of the first regret evaluation and the first complacency evaluation,

$$|u_1(x)| = |u_3^*(x^*)|$$

holds, i. e.

$$|u_1(x)| = |u_3^* \circ \theta(x)|$$

Furthermore, for every $x \in P$, $u_1(x) \leqq 0$ and $u_3(x^*) \geqq 0$. Thus

$$u_1(x) = -u_3^* \circ \theta(x)$$

From which (1) follows in the case of $n=1$ and $m=3$.

For the case of $n=2$, and $m=4$, the proof is similar, because for every $x \in P$, $|u_2(x)| = |u_4^*(x^*)|$.

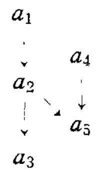
DEFINITION 2. 1. Two evaluations $u: (P, \gtrsim) \to (\mathbf{N}, \geqq)$ and $v: (P, \gtrsim) \to (\mathbf{N}, \geqq)$ are said to be *equivalent* if the mapping, for every $x \in P$,

$$u(x) \longrightarrow v(x)$$

is a preorder-isomorphism of $(\{u(x) | x \in P, \}, \geqq)$ onto $(\{v(x) | x \in P\}, \geqq)$.

PROPOSITION 2. 2. (1) *The first regret evaluation and the second regret evaluation are not always equivalent, and* (2) *The first complacency evaluation and the second complacency evaluation are also not always equivalent.*

Proof. Let a set $\{a_1, a_2, a_3, a_4, a_5\}$ be represented by the arrow diagram



Then $u_1(a_3) = u_1(a_5) = -2$, but $-2 = u_2(a_3) \neq u_2(a_5) = -3$

(2) The second part follows from the duality principle.

PROPOSITION 2. 3. *Let* $|P|$ *denote the number of elements of P. If* $|P| \leqq 2$,

(1) *the first regret evaluation and the second regret evaluation are equivalent, and*

(2) *the first complacency evaluation and the second complacency evaluation are equivalent.*

Proof. (1) If $|P|=1$, the proof is obvious.

Suppose $P = \{a_1, a_2\}$. Then, P is preordered in one of the following ways:

$$a_1 \longrightarrow a_2, \quad a_1 \longleftrightarrow a_2, \quad a_1 \not\longleftrightarrow a_2.$$

Hence, in whichever way P is preordered,

$$u_1(a_1) = u_2(a_1) \text{ and } u_1(a_2) = u_2(a_2)$$

Therefore, the two evaluations are eqivalent.

(2) The second part can be proved in an analogous way.

PROPOSITION 2.4. *Let* $|P| > 2$. *If* P *is completely ordered, then the first regret evaluation and the second regret evaluation are equivalent, and likewise the first complacency evalution, and the second complacency evaluation are equivalent.*

Proof. For any element $x \in P$, $u_1(x) = u_2(x)$ and $u_3(x) = u_4(x)$ hold. Q. E. D.

PROPOSITION 2.5. *The first regret evaluation is not always equivalent to the first complacency evaluation.*

Proof. Suppose there exist two disjoint chains

$$C_1: \ x_1 > x_2 > \cdots\cdots\cdots\cdots > x_m$$

$$C_2: \ y_1 > y_2 > \cdots\cdots\cdots\cdots > y_n$$

in P such that m is smaller than n, and $(C_1, \gtrsim)$ and $(C_2, \gtrsim)$ are direct summands of $(P, \gtrsim)$. Then, $u_1(x_m) > u_1(y_n)$, but $u_3(x_m) = u_3(y_n)$. Therefore, the two evaluations are not equivalent.

PROPOSITION 2.6. *If there exists a positive integer* α *such that* P *is the direct sum of connected preosets, each having an* $(1, \alpha)$*-span, then the first regret evaluation and the first complacency evaluation are equivalent.*

Proof. Suppose the condition is satisfied.

Then, for every $x \in P$

$$u_3(x) = u_1(x) + \alpha.$$

Therefore, for every x, y in P,

$$u_1(x) \geqq u_1(y) \iff u_3(x) \geqq u_3(y).$$

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