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DECOMPOSABLE OPERATORS ON THE DIRECT SUM OF BANACH SPACES

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[I] C. Foias introduced the notion of the decomposable operator with a help of spectral maximal spaces in the following way:

DEFINITION 1.1. A closed vector subspace D of a Banach space X is called *spectral maximal space for* $T\in B(X)$ if,

(i) D is invariant under T,

(ii) If S is another closed vector subspace of X invariant under T such that $\sigma(T|S)\subset\sigma(T|D)$, then $S\subset D$.

DEFINITION 1.2. An operator $T\in B(X)$ is *decomposable* if, for every open covering $\{G_i\}_{i=1}^n$ of $\sigma(T)$, there exists a system $\{D_i\}_{i=1}^n$ of spectral maximal spaces for T such that

(1) $\sigma(T|D_i)\subset G_i$ $(i=1, 2, \cdots, n)$ and

(2) $X=\sum_{i=1}^n D_i$

It is known that if a projection $P_\gamma : X\to X_\gamma$ such that the range of P_γ is X_γ for each $\gamma\in\Gamma$ i.e. $R(P_\gamma)=X_\gamma$, $\gamma\in\Gamma$, where $X=\bigoplus_{\gamma\in\Gamma}X_\gamma$, then we have the following properties:

(i) $\sum_{\gamma\in\Gamma}P_\gamma=I$ (I is an identity operator)

(ii) $P_\alpha P_\beta=\delta_{\alpha\beta}P_\alpha$ ($\delta_{\alpha\beta}$ is the kronecker's delta)

(iii) A restriction of T on X_γ is T_γ, and

(iv) $TP_\gamma=P_\gamma T$ for each $\gamma\in\Gamma$.

If the indexed set Γ is finite, a spectrum $\sigma(T)$ is represented by $\sigma(T)=\bigcup_{\gamma\in\Gamma}\sigma(T_\gamma)$.

In this note Γ is assumed to be always finite. Under these circumstances, we shall prove the following

THEOREM 1. *Suppose that* $X=\bigoplus_{\gamma\in\Gamma}X_\gamma$, $R(P_\gamma)=X_\gamma$, $T\in B(X)$ *and* $T=\bigoplus_{\gamma\in\Gamma}T_\gamma$, *then* T *is decomposable if and only if each* T_γ *is decomposable.*

Proof. Suppose T is decomposable, and let $V^{(\gamma)}$ be any finite open cover of $\sigma(T_\gamma)$ for each $\gamma\in\Gamma$. The collection $V=\{V^{(\gamma)}\}_{\gamma\in\Gamma}$ is certainly a finite open cover of the spectrum $\sigma(T)$ since $\sigma(T)=\bigcup_{\gamma\in\Gamma}\sigma(T_\gamma)$.

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According to the assumption, there exist spectral maximal spaces corresponding to the open cover V, we denote it by $\{D_Q\}_{Q\in V}$ such that $\sigma(T|D_Q)\subset Q$ and that $X=\sum_{Q\in V} D_Q$.

Observing the fact that

$$\sigma(T_\gamma|D_Q)=\sigma(T|X_\gamma\cap D_Q)$$

and

$$\sigma(T_\gamma|D_Q)\subset\sigma(T|D_Q).$$

We have

(1.1) $$\sigma(T_\gamma|D_Q^{(\gamma)})\subset Q \text{ where } D_Q^{(\gamma)}=X_\gamma\cap D_Q,\ \gamma\in\Gamma.$$

Since D_Q is a spectral maximal space for T, it is easily shown that $D_Q^{(\gamma)}$ is also a spectral maximal space of T_γ, $\gamma\in\Gamma$.

To complete the proof that an operator T_γ $(\gamma\in\Gamma)$ is decomposable, we have to show

$$X_\gamma=\sum_{Q\in V^{(\gamma)}} D_Q^{(\gamma)}.$$

This equality follows from the following facts: since Γ is finite, we may write

$$\{V^{(\gamma_i)}\}_{\gamma_i\in\Gamma}\qquad(i=1,\ 2,\cdots,n)$$

instead of $\{V^\gamma\}_{\gamma\in\Gamma}$, and also $x=(x_\gamma)=(x_{\gamma_1},x_{\gamma_2},\cdots,\ x_{\gamma_n})$.

Whence each element $x\in X$ can be represented in the following two ways, namely

$$x=\sum_{i=1}^{n} x_{\gamma_i}$$

and

$$x=\sum_{Q\in V} x_Q=\sum_{Q\in\{V^{(\gamma_i)}\}_{i=1}^{n}} x_Q,\quad x_Q\in D_Q.$$

Therefore we have

$$x_{\gamma_i}=\sum_{Q\in\{V^{(\gamma_i)}\}_{i=1}^{n}} x_Q,\quad (i=1,\ 2,\cdots,\ n).$$

This means that

$$X_{\gamma_i}=\sum_{Q\in V^{(\gamma_i)}} D_Q^{(\gamma_i)},\ \text{that is,}\ X_\gamma=\sum_{Q\in V^{(\gamma)}} D_Q^{(\gamma)}\quad(\gamma\in\Gamma).$$

Conversely, we assume an operator $T_\gamma\in B(X_\gamma)$ is decomposable for each $\gamma\in\Gamma$, and let V be any finite open cover of $\sigma(T)$. We can choose a finite open cover $V^{(\gamma)}$ of $\sigma(T_\gamma)$ for each $\gamma\in\Gamma$ such that

$$V^{(\gamma)}\subset V \text{ and } \bigcup_{\gamma\in\Gamma} V^{(\gamma)}=V,$$

this is possible since $\sigma(T)=\bigcup_{\gamma\in\Gamma}\sigma(T_\gamma)$.

By assumption, there exists a corresponding family of spectral maximal spaces $\{D_Q^{(\gamma)}\}_{Q\in V^{(\gamma)}}$ with

(1.2) $$\sigma(T_\gamma|D_Q^{(\gamma)}\subset Q \text{ for each } Q\in V^{(\gamma)}$$

with $X_1=\sum_{Q\in V^{(\gamma)}} D_Q^{(\gamma)}$, $\gamma\in\Gamma$. Therefore we have

(1.3) $$X=\bigoplus_{\gamma\in\Gamma} X_\gamma\equiv\sum_{\gamma\in\Gamma} X_\gamma=\sum_{\gamma\in\Gamma}\sum_{Q\in V^{(\gamma)}} D_Q^{(\gamma)}=\sum_{Q\in\{V^{(\gamma)}\}_{\gamma\in\Gamma}} D_Q^{(\gamma)}.$$

Observing the relations

$$D_Q^{(\gamma)} \subset X_\gamma \quad \text{and} \quad T_\gamma = T | X_\gamma$$

we get

$$\sigma(T_\gamma | D_Q^{(\gamma)}) = \sigma(T | D_Q^{(\gamma)}) \subset Q \text{ for each } Q \in \{V^{(\gamma)}\}_{\gamma \in \Gamma}. \tag{1.4}$$

The assertions (1.3) and (1.4) imply that the operator T is decomposable. ∎

[II] The Dunford integral of a decomposable operator.

Main purposes of this section are to show that under what conditions $f(T)$ is decomposable?, and what relationships are there the decomposability of $f(T)$ and of $f_\gamma(T_\gamma)$?

In order to solve these questions, author used results which were proved by J. D. Gray (see [8]).

By sets $F(T)$ and $F(x, T)$ we mean a classes of all analytic functions in some neighborhood of $\sigma(T)$ and $\sigma(x, T)$ respectively.

DEFINITION 2.1. An open set U_x is said to be *T-admissible* at x if

(i) $\sigma(x, T) \subseteq U_x$

(ii) U_x consists of only a finite number of components

(iii) a boundary of U_x, denote it by ∂U_x, consists of finite number of closed disjoint, rectifiable Jordan curves.

For each $f \in F(x, T)$, $f(T)x$ is defined by the integral

$$f(T)x = \frac{1}{2\pi i} \int_{\partial U_x} U(\zeta, x, T) f(\zeta) d\zeta \tag{2.1}$$

where $U(\cdot, x, T) : \rho(x, T) \to X$ is the extended resolvent of T at x.

LEMMA 1. *A vector subspace $D \subset X$ is invariant under $T \in B(X)$ if and only if D is invariant under $f(T)$, $f \in F(T)$.*

The proof can be done using operational calculus and polynomial approximation, we omit the detailed calculations.

LEMMA 2. *A necessary and sufficient condition of a closed subspace D of X is a spectral maximal space under $f(T)$ is that D is a spectral maximal space under T.*

Proof. Let S be another invariant subspace of X under $f(T)$ that is $f(T)S \subset S$ and assume that

$$\sigma(f(T) | S) \subset \sigma(f(T) | D). \tag{2.2}$$

We need only to show that (2.2) is true if and only if $\sigma(T|S) \subset \sigma(T|D)$.
According to the spectral mapping theorem and the fact that $\sigma(T) = \bigcup_{x \in X} \sigma(x, T)$, we have

$$\begin{aligned} f(\sigma(T|S)) &= f(\bigcup_{x \in S} \sigma(x, T)) = \bigcup_{x \in S} f(\sigma(x, T)) \\ &= \bigcup_{x \in S} \sigma(x, f(T)) = \sigma(f(T) | S) \\ &= \sigma(f(T|S)). \end{aligned}$$

Thus, by (2.2), it follows that

$$f(\sigma(T|S)) \subset f(\sigma(T|D)).$$

Hence we have

$$\sigma(T|S) \subset \sigma(T|D). \tag{2.3}$$

Conversely (2.3) implies (2.2), this completes the proof. ■

Now we have prepared to prove the following

THEOREM 2. *Suppose that $f \in F(T)$ is not a constant on some neighborhood of $\sigma(T)$, that is $f' \neq 0$ on some neighborhood of $\sigma(T)$, then the operator T is decomposable if and only if its Dunford integral $f(T)$ is decomposable.*

Proof. Assume that an operator T is decomposable. Since $f(T) \in B(X)$, $\sigma(f(T))$ is bounded closed in complex plane C with usual topology. Therefore, in any open covering of $\sigma(f(T))$, there exists a finite open cover $\{O_i\}_{i=1}^n$ that is $\sigma(f(T)) \subset \bigcup_{i=1}^n O_i$, whence $f(\sigma(T)) \subset \bigcup_{i=1}^n O_i$ by the spectral mapping theorem.

Therefore

$$\sigma(T) \subset f^{-1}(\bigcup_{i=1}^n O_i) = \bigcup_{i=1}^n f^{-1}(O_i) = \bigcup_{i=1}^n G_i, \tag{2.4}$$

obviously $f^{-1}(O_i) = G_i$ $(i=1, 2, \cdots, n)$ is open since f is analytic and so continuous.

For each $Z_0 \in \sigma(T)$ we may choose $\varepsilon(Z_0) > 0$ so that f is locally one-to-one and open on $S(Z_0, \varepsilon(Z_0))$ and together with the assumption, there exists an inverse function f^{-1} on some neighborhood of $f(S(Z_0, \varepsilon(Z_0)))$ by the inverse function theorem in complex analysis.

Since $\sigma(T) \subset \bigcup_{Z \in \sigma(T)} S(Z, \varepsilon(Z))$ and $\sigma(T)$ is compact, there exists a finite open cover $\{S(Z_k), \varepsilon(Z_k)\}_{k=1}^m$ namely

$$\sigma(T) \subset \bigcup_{k=1}^m S(Z_k, \varepsilon(Z_k)). \tag{2.5}$$

If we put $S(Z_k, \varepsilon(Z_k)) = S_k$ $(k=1, 2, \cdots, m)$, then $(\bigcup_{i=1}^n G_i) \cap (\bigcup_{k=1}^m S_k) = \bigcup_{i=1}^n \bigcup_{k=1}^m (G_i \cap S_k)$ is an open cover of $\sigma(T)$ and each $f(G_i \cap S_k)$ $(i=1, 2, \cdots, n,\ k=1, 2, \cdots, m)$ is open on which an inverse function f^{-1} exists.

Therefore without loss of generality it may be considered that $f: G_i \to f(G_i)$ is one-to-one continuous and open on G_i $(i=1, 2, \cdots, n)$. Hence $f^{-1}(O_i) = G_i$, $f(G_i) = O_i$ $(i=1, 2, \cdots, n)$.

According to the assumption, there exists a system of spectral maximal spaces $\{D_i\}_{i=1}^n$ corresponding to $\{G_i\}_{i=1}^n$ such that

$$\sigma(T|D_i) \subset G_i\ (i=1, 2, \cdots, n),\ \sum_{i=1}^n D_i = X.$$

Thus we have $f(\sigma(T|D_i)) \subset f(G_i)$ $(i=1, 2, \cdots, n)$ i.e.

$$\sigma(f(T|D_i)) = \sigma(f(T)|D_i) \subset O_i\ (i=1, 2, \cdots, n)$$

and

$$\sum_{i=1}^n D_i = X.$$

This and Lemma 2 show that $f(T)$ is decomposable.

Conversely, suppose that $f(T)$ is decomposable, and let $\{G_i\}_{i=1}^n$ be a finite open cover of $\sigma(T)$, that is, $\sigma(T)\subset\bigcup_{i=1}^n G_i$. By similar arguments as before, we may assume that $f: G_i\to f(G_i)$ is homeomorphic. Thus $\sigma(f(T))\subset\bigcup_{i=1}^n f(G_i)=\bigcup_{i=1}^n O_i$, where $f(G_i)=O_i$ is open for each $i=1,2,\cdots,n$. From the hypothesis of decomposability of $f(T)$, there exists a system of spectral maximal spaces for $f(T)$ such that

$$\sigma(f(T)\,|\,D_i)\subset O_i\,(i=1,2,\cdots,n),\ \sum_{i=1}^n D_i=X.$$

Thus

$$\sigma(f(T)\,|\,D_i)=f(\sigma(T|D_i))\subset O_i\quad(i=1,2,\cdots,n),$$

therefore

$$\sigma(T|D_i)\subset f^{-1}(O_i),\quad \sum_{i=1}^n D_i=X.$$

This shows T is decomposable. ■

The second question in the beginning of this section will be answered in the Theorem 3. For this purpose we have to prove the following Lemmas.

LEMMA 3. *Let* $F(T)$, $F(x,T)$ and $F(x_\gamma,T_\gamma)$ *be sets of all analytic functions on some neighborhoods of* $\sigma(T)$, $\sigma(x,T)$ *and* $\sigma(x_\gamma,T_\gamma)$ *respectively, then we have the following inclusions:*

$$F(T)\subset F(x,T)\subset F(x_\gamma,T_\gamma).$$

Proof. For any $f\in F(T)$, we can choose an open set V in which f is analytic and such that

$$\sigma(x,T)\subset V\subseteq W$$

where f is analytic in W, this is possible since $\sigma(x,T)\subset\sigma(T)$. Thus the restriction $f|V$ in an element of $F(x,T)$. Observing in case that $V=W$, we get $f\in F(x,T)$, that is $F(T)\subset F(xT,)$. Simiarly, we obtain the subsequent inclusion. ■

LEMMA 4. *Let* $\boldsymbol{U}_\gamma(\cdot,x_\gamma,T_\gamma):\rho(x_\gamma,T_\gamma)\to X_\gamma$ *and* $\boldsymbol{U}(\cdot,x,T):\rho(x,T)\to X$ *be extended resolvent operator of* T_γ *at* x_γ *and of* T at x *respectively, then* U_γ *is the analytic extension of* $\boldsymbol{U}|X_\gamma:\rho(x,T)\to X_\gamma$ *over the resolvent set* $\rho(x_\gamma\ T_\gamma)$.

Proof. It is known that $\rho(x,T)\subset\rho(x_\gamma,T_\gamma)$ (see [6]). Since $(\zeta I-T)\,\boldsymbol{U}(\zeta,x,T)=x$,

$$x_\gamma=[(\zeta I-T)\,\boldsymbol{U}(\zeta,x,T)]\,|X_\gamma=(\zeta I_\gamma-T_\gamma)\,[\boldsymbol{U}(\zeta,x,T)\,|X_\gamma]$$

for each $\zeta\in\rho(x,T)$. The single valued extension peroprty of T_γ implies that

$$\boldsymbol{U}(\cdot,x,T)\,|X_\gamma=\boldsymbol{U}_\gamma(\cdot,x_\gamma,T_\gamma)\ \text{ on }\rho(x,T).$$

This completes the proof. ■

From Lemma 3 and Lemma 4, it is easily seen that

$$f(T)\,x\,|X_\gamma=\left[\frac{1}{2\pi i}\int_{\partial U_x}\boldsymbol{U}(\zeta,x,T)\,f(\zeta)\,d\zeta\right]|X_\gamma$$

$$=\frac{1}{2\pi i}\int_{\partial U_x}\boldsymbol{U}(\zeta,x,T)\,|X_\gamma f_\gamma(\zeta)\,d\zeta$$

$$=\frac{1}{2\pi i}\int_{\partial U_{x_\gamma}} U(\zeta, x_\gamma, T_\gamma) f_\gamma(\zeta)\, d\zeta$$
$$=f_\gamma(T_\gamma)\, x_\gamma.$$

Moreover $\sum_{\gamma\in\Gamma}\|f_\gamma(T_\gamma)x_\gamma\|^2<\infty$, thus the vector $f(T)x$ can be represented by a direct sum of the family $\{f_\gamma(T_\gamma)x_\gamma\}_{\gamma\in\Gamma}$, namely $f(T)x=\bigoplus_{\gamma\in\Gamma} f_\gamma(T_\gamma)x_\gamma\equiv\sum_{\gamma\in\Gamma} f_\gamma(T_\gamma)x_\gamma$ where the index set Γ is finite.

Now we are in the position to prove the following

THEOREM 3. *Let an index set Γ be finite. Suppose that*

$$X=\bigoplus_{\gamma\in\Gamma} X_\gamma,\quad T\in B(X),\quad P_\gamma T=T_\gamma,\quad R(P_\gamma)=X_\gamma\quad (\gamma\in\Gamma)$$

and that $f'\neq 0$ on some neighborhood of $\sigma(T)$. Then $f(T)$ is decomposable if and only if $f_\gamma(T_\gamma)$ is decomposable for each $\gamma\in\Gamma$.

Proof. The spectral radius $r_\sigma(T_\gamma)$ is less than $r_\sigma(T)$ for each $\gamma\in\Gamma$, i.e., $r_\sigma(T_\gamma)\leqslant r_\sigma(T)$, since $\sigma(T)=\bigcup_{\gamma\in\Gamma}\sigma(T_\gamma)$ and $r_\sigma(T)=\sup_{z\in\sigma(T)}|z|$. The assumption that f' does not vanish on some neighborhood of $\sigma(T)$ implies that f_γ' can not be 0 on some neighborhood of $\sigma(T_\gamma)$ for each $\gamma\in\Gamma$. Theorem 2 showed that T_γ is decomposable if and only if $f_\gamma(T_\gamma)$ is decomposable for each $\gamma\in\Gamma$. Moreover, we knew that an operator $T\in B(X)$ is decomposable if and only if $T_\gamma\in B(X_\gamma)$ is decomposable for each $\gamma\in\Gamma$ (Theorem 1). Thus we have the following diagram:

$$\begin{array}{ccc} T_\gamma \text{ is decomposable for each } \gamma\in\Gamma & \rightleftarrows & T \text{ is docomposable} \\ \updownarrow & & \updownarrow \\ f_\gamma(T_\gamma) \text{ is decomposable for each } \gamma\in\Gamma & \rightleftarrows & f(T) \text{ is decomposable} \end{array}$$

Therefore we obtain the required conclusion that $f(T)$ is decomposable if and only if $f_\gamma(T_\gamma)$ is decomposable for each $\gamma\in\Gamma$. ∎

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