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STRONG AND WEAK TOPOLOGIES ON GENERALIZED INNER PRODUCT SPACES

BY

Kim, DONG HWA
Korea University, Seoul, KOREA

1. INTRODUCTION

In the present note we enumerate and derive some fundamental properties of strong and weak topologies on generalized inner product spaces (GIP) which E. Prugovečki introduced.

In this paper, we will show that topologies in a GIP space can also be defined by the family of semi-norms.

By notion of semi-norms, we can simplify the proof on some fundamental properties can we can obtain the sharpend results of metrizablity.

we show that if $\mathfrak{A}$ is a finite set in a GIP space$(\mathfrak{L}, \mathfrak{A}, \mathfrak{N})$ with strong topologies, then GIP space $(\mathfrak{L}, \mathfrak{A}, \mathfrak{N})$ is normable.

2. GENERALIZED INNER PRODUCT SPACES

An inner product space is a linear spaces on which an inner(scalar) product(x,y) is defined. When the linear space is complex, we adapt the convention that(x,y) is anti-linear with respect to the first argument, and consequently linear with respect to the second argument.

Definition 2.1 A linear space $\mathfrak{L}$ is a generalized inner product (GIP) if and only if;

1. There is a subspace $\mathfrak{N}$ of $\mathfrak{L}$ which is an inner product space (which will be called the nucleus of the GIP space);
2. There is a set α of linear operators on $\mathfrak{L}$ which is adequate with respect to $\mathfrak{N}$, i.e., it has the following properties
   (a) Each element of $\mathfrak{N}$ maps $\mathfrak{L}$ into $\mathfrak{N}$ i.e. $\mathfrak{A}\mathfrak{L}\subset\mathfrak{N}$;
   (b) The relation $Ax=0$ is satisfied for all $A\in\mathfrak{A}$ only by $x=0$
   (c) We denote such a GIP space by the triple$(\mathfrak{L},\mathfrak{A},\mathfrak{N})$

Example 1. Every inner product space is a GIP space in a trivial sense, i.e., $\mathfrak{A}=\mathfrak{L}$ and $\mathfrak{A}=\{1\}$ where 1 denote the identity operator on $\mathfrak{L}$.

Example 2. $\mathfrak{L}$ is the family of all one-row infinite matrices with real elements$(a_1, a_2, \cdots\cdots\cdots)$, and take $\mathfrak{N}$ to be the one-dimensional space of all one row real metrices$(a_1, 0, 0, \cdots\cdots\cdots)$ in which only the first element is nonvanishing. Adapt the inner product in $\mathfrak{N}$ to be

$$((a_1, 0, 0, \cdots\cdots)(b_1, 0, 0, \cdots\cdots)) = (a_1, 0, 0, \cdots\cdots)\begin{pmatrix} b_1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

If we choose

$$\mathfrak{N} = \{p_1, p_2, \cdots\cdots\}$$

where p_n is the linear operator

$$P_n(a_1, a_2, \cdots\cdots, \ a_n\cdots\cdots) = (a_n, 0, 0, \cdots\cdots)$$

then $(\mathfrak{L}, \mathfrak{A}, \mathfrak{N})$ constitute a GIP space.

Example 3. $\mathfrak{L}$ is the family of all real continuous functions on the real line.

Let us choose the nucleus to of all square integrable functions in $\mathfrak{L}$, and adapt the inner product in $\mathfrak{N}$ to be

$$(x, y) = \int_{-\infty}^{\infty} x(t) \cdot y(t)\, dt$$

Take $\mathfrak{A}$ to be the family of all projectors $E(I)$,

$$(E(I)x)(t) = \chi_I(t)\, x(t)$$

($\chi_I(t)$ denotes the characteristic function of the I) corresponding to all the finite non-degenerate intervals, then $(\mathfrak{L}, \mathfrak{A}, \mathfrak{N})$ is a GIP space.

3. STRONG TOPOLOGIES

As to a GIP space of use in quantum mechanics, we must choose the topologies which make GIP spaces into a topological Vector space. We shall introduce in GIP spaces strong topologies which was obtained by E. Prugovečki (3).

Strong topologies in GIP spaces is constructed by the neighborhood bases of at $0 \in \mathfrak{L}$ from sets of the form

$$V(0; A_1, A_2, \cdots\cdots, \ A_n; \varepsilon) = \{x;\ \| A_1 x \| < \varepsilon, \cdots\cdots, \ \| A_n x \| < \varepsilon, x \in \mathfrak{L}\}$$

for all $\varepsilon > 0$, $A_1, A_2, \cdots\cdots, A_n \in \mathfrak{A}$ and $n = 1, 2, \cdots\cdots$.

However, strong topologies which E. Prugorečki introduced in GIP spaces can be defined by the semi-norms.

In a given GIP space $(\mathfrak{L}, \mathfrak{A}, \mathfrak{N})$, We can assign to each $A \in \mathfrak{A}$ a map:

$$q_x(A);\, x \rightarrow \| Ax \|$$

Then it is easy to see that $q_x(A)$ define a semi-norm on $\mathfrak{L}$. One way of constructing a locally convex topology is described in the following lemma (1).

Lemma 2.1 Suppose that X is a linear space and that U is a nonempty family of nonempty subsets of X with the properties.

1. Each member of u is balanced convex and absorbing.
2. If $u \in U$, there exists some α such that $0 < \alpha \leqq \frac{1}{2}$ and $\alpha u \in U$.
3. If $u_1 \in U$ and $u_2 \in U$, there exists $u_3 \in U$ such that $u_3 \subset u_1 \cap u_2$.
4. If $u \in U$ and $x \in u$, there exists $v \in U$ such that $x + V \subset U$. Then there is a unique locally convex topology with U as a base at o for X

Let us denote the family $\{q(A); A \in \mathfrak{A}\}$ by $q\{\mathfrak{A}\}$. Then it follows from proceeding lemma that $q\{\mathfrak{A}\}$ define locally convex topology on the GIP space $(\mathfrak{L}, \mathfrak{A}, \mathfrak{N})$, and morever it is easy to see that the topology generated by $q\{\mathfrak{N}\}$ is the strong topology in a GIP space $(\mathfrak{L}, \mathfrak{A}, \mathfrak{N})$.

Next we will show that a GIP space with strong topologies is Hausdorff.

It is well known that if X is a locally convex topological linear space with topology generated by a family p of semi-norms, then X is a Hausdorff-space if and only if $p(x)=0$ for each $p \in P$ implies $x=0$.

Theorm 3.1 A GIP space $(\mathfrak{L}, \mathfrak{A}, \mathfrak{N})$ with strong topology is Hausdorff.

Proof: Let us $q_x(A)=0$ for all $q(A) \in q\{\mathfrak{A}\}$, then $(Ax, Ax)=0$ for all $A \in \mathfrak{A}a$ from elementary properties of inner product space. As $Ax=0$ is true for all $A \in \mathfrak{A}$ from properties of the inner product. It follows from definition 2.1 that $x=0$.

In settling the important question of completion, it is very convenient when a topological vector space is metrizable.

E. Prugovečki proved the metrizability of a GIP space $(\mathfrak{L}, \mathfrak{A}, \mathfrak{N})$ with the strong topology for the following situation.

A GIP Space $(\mathfrak{L}, \mathfrak{A}, \mathfrak{N})$ with strong topology is metrizable if there is a countable subset $\mathfrak{B}$ of $\mathfrak{A}$ which has the property that for any $A \in \mathfrak{A}$ there is a B in the linear manifold L_B generated by B such that

$$\|Bx\| \geqq \|Ax\| \qquad \text{for all } x \in \mathfrak{L}$$

In the following, We will sharpen the proceeding results.

Theorem 3.2 Let A be a linear operator on GIP space $(\mathfrak{L}, \mathfrak{B}, \mathfrak{N})$ with strong topology into inner product space $\mathfrak{N}$ which has the property that for any $A \in \mathfrak{A}$ there is a B in the linear manifold L_B generated by B, such that

$$\|Bx\| \geqq \|Ax\| \qquad \text{for all } x \in \mathfrak{L}$$

then A is continuous.

Proof: Let N be ε-neighborhood of o in $\mathfrak{N}$. Then we can find a $B \in L_B$ by $\|Bx\| \geqq \|Ax\|$.

As $\mathfrak{B}$ generates L_B, We have that

$$B=\lambda_1 B_1+\cdots\cdots+\lambda_K B_K, \qquad B_1, \cdots\cdots B_K \in \mathfrak{B}$$

and consequently

$$q_x(B) \leqq |\lambda_1| q_x(B_1)+\cdots\cdots+|\lambda_K| q_x(B_K)$$

for all $x \in \mathfrak{L}$. There exists a positive such that

$$\delta \leqq \operatorname{Min}\left(\frac{\varepsilon}{k|\lambda_1|}, \cdots\cdots, \frac{\varepsilon}{k|\lambda_K|}\right),$$

hence a set $\{x;\ q_x(B_1)<\delta, \cdots\cdots,\ q_x(B_K)<\delta\}$ is a neighborhood at o in $(\mathfrak{L}, \mathfrak{B}, \mathfrak{N})$.

If $x \in \{x;\ q_x(B_1)<\delta_1, \cdots\cdots,\ q_x(B_K)<\delta\}$, then

$$\|Ax\| \leqq \|Bx\| < \varepsilon$$

So the proof is complete.

From the above theorem 3.1, We can assure that result by E. Prugovečki is sharpend by the following theorem.

Theorem 3.3. A GIP space $(\mathfrak{L}, \mathfrak{A}, \mathfrak{N})$ with strong topology is metrizable if there is a countable subset $\mathfrak{B}$ of $\mathfrak{A}$ which has the property that $\mathfrak{A}$ consists of continuous linear operator on a GIP space ($\mathfrak{L}$, $\mathfrak{B}$, $\mathfrak{N}$) with strong topology into $\mathfrak{N}$.

Proof: The family of all sets

$$\{x;\ q_x(A) < \varepsilon,\ A \in \mathfrak{A}, \varepsilon > 0\}$$

form a neighborhood basis of 0 in ($\mathfrak{L}$, $\mathfrak{A}$, $\mathfrak{N}$) with strong topology. Let N be the ε-neighborhood of o in n. Then $\{x;\ q_x(A) < \varepsilon\} = \{x;\ \|Ax\| < \varepsilon\} = A^{-1}(N)$ and A is continuous linear operator on ($\mathfrak{L}$, $\mathfrak{B}$, $\mathfrak{N}$) into $\mathfrak{N}$. Hence $\{x;\ q_x(A) < \varepsilon\}$ is a member of strong topology of ($\mathfrak{L}$, $\mathfrak{B}$, $\mathfrak{N}$).

Therefore the two topologies are equivalent. As the strong topology of ($\mathfrak{L}$, $\mathfrak{B}$, $\mathfrak{N}$) is Hausdorff and is generated by countable family of semi-norms, ($\mathfrak{L}$, $\mathfrak{B}$, $\mathfrak{N}$) with strong topology is metrizable. So GIP space ($\mathfrak{L}$, $\mathfrak{N}$, $\mathfrak{N}$) is metrizable.

Next we will consider normability of a GIP space. If a locally convex Space is normable, it is metrizable, but the converse is not true.

Theorem 3.4. The GIP space ($\mathfrak{L}$, $\mathfrak{A}$, $\mathfrak{N}$) with strong topology is normable if $\mathfrak{A}$ is a finite set.

Proof; Let $\mathfrak{A}$ consist of $A_1, A_2, \cdots\cdots A_n$,

$$q(x) = \max_{1 \leqq i \leqq n} q_x(A_i)$$

is continuous semi-norm of ($\mathfrak{L}$, $\mathfrak{N}$, $\mathfrak{A}$) and $q_x(A_i) \leqq q(x)$.

Hence the strong topology of ($\mathfrak{L}$, $\mathfrak{A}$, $\mathfrak{N}$) can also be defined by a single semi-norm q. Furthermore, ($\mathfrak{L}$, $\mathfrak{A}$, $\mathfrak{N}$) s Hausdorff. Therefore q(x) is norm.

4. WEAK TOPOLOGIES

If ($\mathfrak{L}$, $\mathfrak{A}$, $\mathfrak{N}$) is a GIP space, We can assign to each $A \in \mathfrak{A}$ and each $\xi \in \mathfrak{N}$ a linear functiona

$$\phi_A \cdot \xi(x) = (\xi, Ax)$$

on $\mathfrak{L}$.

Then the linear space $\mathfrak{M}$ (over the same scalar field as $\mathfrak{L}$) spanned by $\mathfrak{M}_o = \{\phi_A \cdot \xi\}$ and the linear space $\mathfrak{L}$ constitute a dual pair (3). We call the weak topology in a GIP space the topology that constructs neighborhood basis of o in $\mathfrak{L}$ from the family of all sets

$$W\{x;\ \phi_1, \phi_2, \cdots\cdots, \phi_n; \varepsilon\} = \{x;\ |\phi_1(x)| < \varepsilon, |\phi_2(x)| < \varepsilon, \cdots\cdots, |\phi_n(x)| < \varepsilon\}$$

corresponding to all $\phi_1, \cdots\cdots, \phi_n \in \mathfrak{M}$, $\varepsilon < 0$.

It is easy to verify that the space $\mathfrak{L}$ provided with the weak topology is locally convex Hausdorff Space.

Theorem 4.1 Each $\phi \in \mathfrak{M}$ is continuous on the vector space with the strong topologies.

Proof: $|\phi_A \cdot \xi(x)| = |(\xi, Ax) \leqq \|\xi\|\ \|Ax\| = \|\xi\|\, q_x(A) < \varepsilon \qquad \left(q_A(x) < \frac{\varepsilon}{\|\xi\|}\right)$

$\|\xi\|\, q_x(A)$ is continuous semi-norm on $\mathfrak{L}$ with the strong topology. Hence the proof is complete.

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