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ON NORMALITY OF SOME PRODUCT SPACES.

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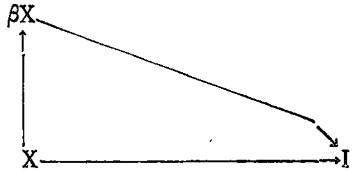
Introduction. In this paper, all spaces are assumed to be Hausdorff. Sorgenfrey's example of a pair of normal spaces whose product is not normal is well known. we will discuss here subsequent work on the problem of suitably restricting spaces X and Y to make $X\times Y$ normal. By giving some particular conditions to Y, if $X\times Y$ is normal, we will note the fact that can take place with the Stone-Ĉech compactification βX of X. Now the following will be showed:

(1) $X\times Y$ is normal if and only if X is Tychonoff space and (Y, h) is a compactification of X such that $h(X)$ is C^*-embedded n Y.

(2) If $X\times I$ is normal for any (Hausdorff) space X there is a continuos $F: \beta X\to I$ such that



commutes.

The central useful fact about the Stone-Ĉech compactification is an extension property, given by the following lemma 1.

DEFINITION. If, for each $\alpha\in A$, $f_\alpha : X\to X_\alpha$, then the evaluation map $e : X\to \prod X_\alpha$ induced by the collection $\{f_\alpha \mid \alpha\in A\}$ is defined as follows for each $x\in X$, $[e(x)]_\alpha = f_\alpha(x)$. That is, for $x\in X$, $e(X)$ is the point in $\prod X_\alpha$ whose αth coordinate is $f_\alpha(x)$ for each $\alpha\in A$.

LEMMA 1. If K is a compact Hausdorff space and $f: X\to K$ is continuous, there is a continuous $F: \beta X\to K$ such that $F\circ e = f$.

In binormal space lemma 2 is well known. This can be used to prove lemma 3.

LEMMA 2. Let X be normal space. The following are then equivalent:

a) X is countably paracompact,

b) each countable open cover $\{U_n \mid n=1, 2, \cdots\}$ of X is shrinkable,

c) each sequence $F_1\supset F_2\supset\cdots\cdots$ of closed sets with empty intersection has an expansion to open sets $G_i\supset F_i$ with $\bigcap G_i=\phi$.

COROLLARY. Let X be Lindelöf space. then X is countably paracompact if and only if X is metacompact.

LEMMA 3. Let X be Hausdorff space. then $X \times I$ is *normal if and only if* X is binormal.

Now let X be a Tychonoff space. If (Y, h) is a compactification of X such that h(X) is C^*-embedded in Y, since (Y, h) is the Stone-Cech compactification of X, we will conclude the fact that $X \times Y$ is normal. We will take the time now to present the following theorem; it is interesting for other reason also.

THEOREM 1. Let X be a Tychonoff space. If (Y, h) is a compactification of X such that h(X) is C^*-embedded in Y, the followings are equivalent:

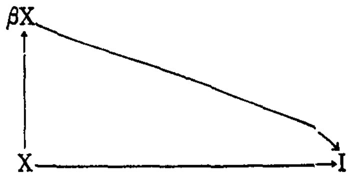
a) $X \times Y$ is nosmal,

b) for each closed set $F \subset Y - X$, there is a locally finite open cover $\{U_\lambda | \lambda \in \Lambda\}$ of X such that $(Cl_Y U_\lambda) \cap F = \phi$, for each $\lambda \in \Lambda$,

c) X is paracompact.

Note the reference about the proof above. let X be a Hausdorff space. If $X \times I$ is normal, the extension property about the Stone-Čech compactification βX of X will be satisfied. that is :

THEOREM 2. If $X \times I$ is normal for any (Hausdorff) space X, there is a continuous $F : \beta X \to I$ such that



commutes.

Proof. Suppose $X \times I$ is normal. Clealry X will be normal. But from Lemma 3 X is binormal. Then each sequence $H_1 \supset H_2 \supset \cdots\cdots$ of closed sets with empty intersection has an expansion to open sets $V_n \supset H_n$ with $\cap V_n = \phi$. Also since X is normal, there is a sequence $F_1, F_2, F_3 \cdots\cdots$ of closed sets with

$$\cap F_n = \phi, \ V_n \supset F_n \supset H_n.$$

let $W_n = X - F_n$. Let A be the complement in $X \times I$ of

$$[W_1 \times [0, 1)] \cup [W_2 \times [0, \tfrac{1}{2})] \cup \cdots\cdots$$

and let $B = X \times \{0\}$. Then A and B are disjoint closed sets in $X \times I$, so there is a Urysohn function $f : X \times I \to I$ with $f(A) = 0$ and $f(B) = 1$.

Let g be the restriction of f to X, then g is continuous on X. By virtue of Lemma 1 there is a continuous extension F of g which carries βX into I.

Reference

Stephen Willard, General topology, Addison-wesley, 1970.