Microeconomic Foundations of Monetary Dynamic Models

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I. Following Harry Johnson's suggestion [1] that the monetary theory be incorporated into the current literature of growth models, there have been many such attempts beginning with Tobin's 1965 *Econometrica* article ([2], [6], [7], [8], [9]).

In most of those macroeconomic models, microeconomic foundations are assumed away. For example, the aggregate savings functions and demand for "money" are assumed to be functions of certain arguments such as per capita capital stock, expected rate of inflation, per capita disposable income and so on.[1] Furthermore the signs of the first-order derivatives of those functions are assumed to be positive or negative with some intuitive explanations.[2] If we look at the microeconomic behavioral analyses of each economic unit, those common sense explanations may not be justified. Thus Sidrauski says, "The major limitation of this analysis is given by the fact that we have postulated a savings function and a demand function for a real cash balances that are not explicitly derived from the maximizing behavior of the individual economic units of the economy."[3]

The purpose of this paper is to show that the functional properties in those aggregate equations can be derived from the microeconomic behavioral analysis rather than assuming them away, and in some cases, the commonly assumed sign properties of those functions may not be justified.

II. We shall begin with a representative behavioral unit. We shall assume that the individual economic unit faces the following situation: he has a constant income flow, w, for his planning horizon $[o, T]$; w will be divided into consumption and savings; the savings will further be divided into an income yielding asset, S, and the non-

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1) See equations (5) and (12) in [1] among many others.

2) Sidrauski's paper [6] is an exception. He sets out the model but does not do what can be done as Jorgenson points out in [4].

3) Sidrauski [7], p. 809.

interest bearing government debt, m, i. e., his asset portfolio consists of S and m; and his initial savings or wealth is given by S_0; S yields a constant interest rate r.

Then the growth path of the income yielding asset is given by

$$\dot{S}_t = rS_t + w - C_t - m_t, \tag{1}$$

where C_t is the consumption of t, and S_0 is given. m_t is the flow of the non-interest bearing asset, and thus if M is the stock of this asset, we have

$$\dot{M}_t = m_t.^{4)} \tag{2}$$

We shall further assume that the holdings of the non-interest bearing government debt, M_t is providing certain services such as time savings in transactions, the transactions cost in converting M_t into S_t and so on. Thus M_t enters into utility function.[5] Then the economic unit has an instantaneous utility function $U(C_t, M_t)$, which is concave with respect to C_t and M_t and twice continuously differentiable. At the end of his planning horizon, he leaves S_T and M_T as bequests, and the bequests yield a certain utility to him, say $Q(S_T, M_T)$. Q is assumed to be in the same class as U function.

Then the representative consumer's problem can be stated as follows:

$$\text{Maximize} \int_0^T U(C_t, M_t)e^{-\alpha t}dt + Q(S_T + M_T)e^{-\alpha T} \tag{3}$$

subject to (1) and (2), where α is the subjective discount rate.

From the first order conditions of the maximization problem, we have

$$\dot{S}_t = rS_t + w - C_t - M_t \tag{4}$$

$$\alpha_1 \dot{C}_t / C_t + \alpha_2 \dot{M}_t / M_t = r - \alpha, \tag{5}$$

where $\alpha_1 = -U_{11}C/U_1$, $\alpha_2 = -U_{12}M/U_1$

$$U_2/U_1 = r \tag{6}$$

$$Q'(S_T + M_T) = U_1(C_T, M_T). \tag{7}$$

All the variables are evaluated at the optimum arc.

Now (6) can be written as

$$U_2(C_t, M_t)/U_1(C_t, M_t) = g(C_t, M_t) = r, \text{ i.e.,}$$

$$g(C_t, M_t) - r = 0. \tag{8}$$

Since $\partial g/\partial C_t \neq 0$, we get $C_t = g^*(M_t, r)$. (9)

From (5) and (9), we have

$$\dot{M}_t = (r - \alpha)/(\alpha_1 g_M^*/g^* + \alpha_2/M_t). \tag{10}$$

4) Later on we shall see that M_0 is endogenously determined.

5) This argument is well established by Patinkin [5] and Sidrauski [6].

Thus the optimum path of the system (4)—(7) can be described by (4), (7), and (10).

Before we consider the system of differential equations (4) and (10), we shall consider

$$g^*_M(M_t,\ r)/g^*(M_t,\ r) \equiv Z(M_t,\ r). \tag{11}$$

Thus we have from (11)

$$\partial Z/\partial M_t = \frac{g^* g^*_{MM} - g^{*2}_M}{g^{*2}} < 0.^{6)} \tag{12}$$

Equations (4) and (10) can be written as

$$\dot{M}_t = F(M_t,\ S_t) \tag{13}$$

$$\dot{S}_t = G(M_t,\ S_t). \tag{14}$$

Consider the steady state solution, i. e., $\dot{M}_t=0$ and $\dot{S}_t=0$.

We have

$$\left.\frac{dS_t}{dM_t}\right|_{\dot{M}_t=0} = -\frac{\partial F}{\partial M} \Big/ \frac{\partial F}{\partial S} \tag{15}$$

$$\left.\frac{dS_t}{dM_t}\right|_{\dot{S}_t=0} = -\frac{\partial G}{\partial M} \Big/ \frac{\partial G}{\partial S}, \tag{16}$$

where

$$\frac{\partial F}{\partial M} = -(r-\alpha)\left(\alpha_1 \frac{\partial Z}{\partial M} - \alpha_2 \frac{1}{M_t^2}\right)/(\alpha_1 Z + \alpha_2/M_t)^2 \tag{17}$$

$$\frac{\partial F}{\partial S} = 0 \tag{18}$$

$$\frac{\partial G}{\partial M} = -\frac{\partial g^*}{\partial M} + \frac{\partial F}{\partial M} \tag{19}$$

$$\frac{\partial G}{\partial S} = r. \tag{20}$$

We have shown that $\frac{\partial Z}{\partial M} < 0$ in (12). Thus we conclude that $\frac{\partial F}{\partial M} > 0$ in (17).[7)]

Case 1: $\left|\frac{\partial g^*}{\partial M}\right| < \frac{\partial F}{\partial M}$.

In this case we know that $\left.\frac{dS_t}{dM_t}\right|_{\dot{M}_t=0} = \infty$ and

$\left.\frac{dS_t}{dM_t}\right|_{\dot{S}_t=0} < 0$ from (15)—(20).

6) From (8) and (9), we have $dC_t = g^*_M\, dM_t$, and thus $\frac{dC_t}{dM_t} = g^*_M$. Thus $g^*_{MM} = \frac{d^2C_t}{dM_t^2} = 0$ because $\frac{dC_t}{dM_t} = \frac{1}{r}$, where r is constant. Note $\frac{U_2}{U_1} = g(C_t,\ M_t) = r$.

7) We are assuming that $r-\alpha>0$, $\alpha_1>0$ and $\alpha_2>0$, i. e., M_t and C_t are complements.

Then the following phase diagram explains the possible time path for M_t and S_t. Of course the time path of M_t determines that of C_t in the sense of equations (8) and (9).

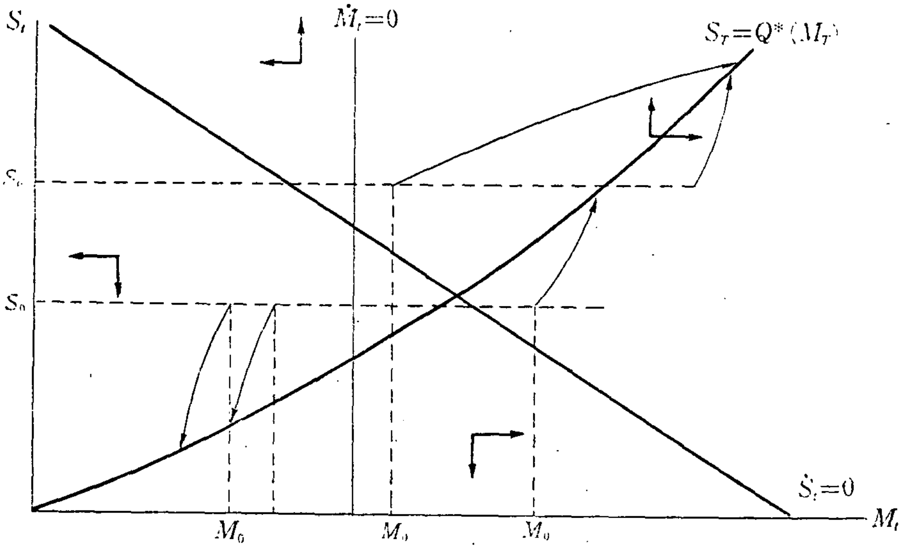

Figure 2

We shall show later on that for each given S_0, M_0 is determined endogenously.

Case 2: $\left|\frac{\partial g^*}{\partial M}\right| > \frac{\partial F}{\partial M}$

In this case we have $\left.\frac{dS_t}{dM_t}\right|_{\dot{S}_t=0} > 0$. Then the optimal time path can be shown by the following phase diagrams.

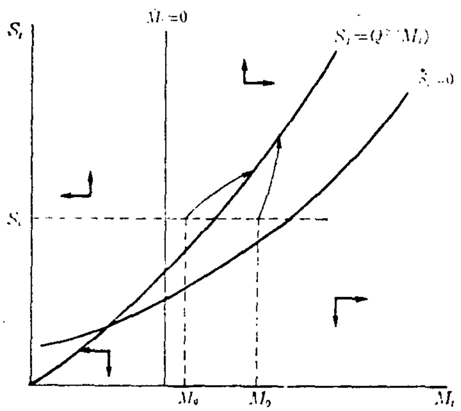

Figure 2-a

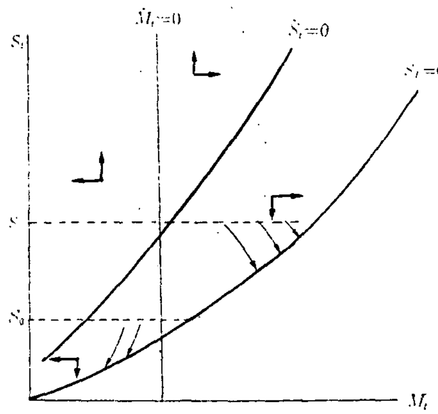

Figure 2-b

In either case the optimal time path depends on M_0 for a given S_0. Now we shall show that M_0 is determined as an endogenous variable in terms of S_0 and all the other parameters. From (4), we have

$$S_T e^{-rT} = S_0 + \int_0^T [w - C(C_{0}, T) - M(M_{0}, T)] e^{-rt} dt \tag{21}$$

and (9) yields

$$C_0 = g^*(M_0, r) \tag{22}$$

Further (7) implies $Q'[S_T + M(C_0, M_0; T)] = U_1(T, C_0, M_0)$ (23)

Now we can see that C_0, M_0 and S_T can be solved in terms of S_0 and all the other parameters of the model in (21), (22) and (23).

We shall consider the following case as an example. Let $U = C^{1-a} M^{1-b}$ and $Q = AW^{1-c}$, where $0 < a < 1$, $0 < b < 1$, $a + b > 1$[8], $0 < c < 1$, A=constant, and $W = C_T + M_T$.

Then we have $$\frac{U_2}{U_1} = \frac{(1-b)}{(1-a)} \frac{C_t}{M_t} = r, \tag{24}$$

$$\text{and } \alpha_1 = a, \ \alpha_2 = (b-1) \tag{25}$$

Thus (5) becomes

$$a \frac{\dot{C}_t}{C_t} + (b-1) \frac{\dot{M}_t}{M_t} = r - \alpha \tag{26}$$

Further (24) implies that $M_t = \frac{(1-b)}{r(1-a)} C_t$ or

$$\dot{M}_t / M_t = \dot{C}_t / C_t \tag{27}$$

Then (26) and (27) provide the following solutions

$$C_t = C_0 e^{\gamma t}, \quad \gamma = \frac{r - \alpha}{a + b - 1}$$

$$M_t = M_0 e^{\gamma t} \tag{28}$$

Now (4) and (28) give

$$S_T e^{-rt} = S_0 + \int_0^T (w - C_0 e^{\gamma t} - \gamma M_0 e^{\gamma t}) e^{-rt} dt \tag{29}$$

And (7) and (28) give

$$B(S_T + M_0 e^{\gamma t})^{-c} = (1-a)(C_0 e^{\gamma t})^{-a} (M_0 e^{\gamma t})^{-b}, \quad B = (1-c)A \tag{30}$$

Further (28) gives

$$M_0 = \frac{(1-b)}{r(1-a)} C_0 \tag{31}$$

Thus (29), (30), (31) will give a solution for C_0, M_0 and S_T in terms of S_0 and all

8) Since $|U| = \begin{vmatrix} U_{11} U_{12} \\ U_{21} U_{22} \end{vmatrix} = (1-a)(1-b) C^{-2a} M^{-2b} (a+b-1)$,
$|U| > 0$ if $a + b > 1$. This makes U concave in C and M.

the other parameters.

III. In the example, we have shown that the optimal time path for C_t and M_t are given by (28). Thus the comparative dynamics properties with respect to r, for example, depend on the signs of $\frac{dC_0}{dr}$ and $\frac{d\gamma}{dr}$, i. e.,

$$\frac{dC_t}{dr}=e^{rt}\frac{dC_0}{dr}+C_0\gamma e^{rt}\frac{d\gamma}{dr}, \tag{32}$$

and

$$\frac{dM_t}{dr}=e^{rt}\frac{dM_0}{dr}+M_0e^{rt}\frac{d\gamma}{dr}. \tag{33}$$

But we know $\frac{d\gamma}{dr}=\frac{1}{a+b-1}>0$. Thus we have to know the signs of $\frac{dC_0}{dr}$ and $\frac{dM_0}{dr}$ in order to determine the signs of $\frac{dC_0}{dr}$ and $\frac{dM_t}{dr}$.

Consider (29), (30) and (31). In (29), we note that

$$S_T=e^{rt}S_0+(w-C_0e^{rt}-\gamma M_0e^{rt})\beta(r), \tag{34}$$

where $\beta(r)=\frac{1}{r}(e^{-rT}-1)$. Further note that $\beta(r)<0$ and

$$\beta'(r)=\frac{1}{r^2}[1-e^{-rT}(1+Tr)]>0.^{9)}$$

We shall define $Q_1(r)=\beta(r)e^{rT}$ and $Q_2(r)=\gamma Q_1(r)$.

Then $Q_1'(r)=\beta'e^{rT}+\gamma'T\beta(r)e^{rT}<0$[10], $\gamma'=\frac{1}{a+b-1}>0$,

and $Q_2'(r)=\gamma'Q_1(r)+\gamma Q_1'(r)<0$ because $Q_1(r)<0$.

Then the total differentiation of (34) gives us

$$dS_T=Q_1(r)dC_0-Q_2(r)dM_0+b_1dr, \tag{35}$$

where $b_1=S_0Te^{rT}+w\beta'-C_0Q_1'-M_0Q_2'$. Now consider equation (30). Solving S_T, we have

$$S_T=-M_0e^{rT}+ZC_0^{\delta}e^{rTS}M_0^{n}e^{rTn},\ Z=\frac{1-a^{-\frac{1}{c}}}{b},\ \delta=\frac{a}{c},\ n=\frac{b-1}{c}.$$

The total differentiation of the above equation provide

$$dS_T=a_{22}dC_0+a_{23}dM_0+b_1dr, \tag{36}$$

where $a_{22}=Ze^{rT(\delta+n)}M_0^{n}\delta C_0^{\delta-1}$, $a_{23}=ZC_0^{\delta}e^{rT(\delta+n)}nM_0^{n-1}-e^{rT}$,

$b_1=ZC_0^{\delta}M_0^{n}T(\delta+n)\gamma'e\gamma^{T}(\delta+n)-M_0T\gamma'e^{rT}$.

Further from the total differentiation of (31), we have

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9) We mean $e^{-rT}(1+Tr)<1$ or $1<e^{rT}-Tr$. If $r=.05$, $T=30$, $rT=1.50$. Thus $e^{rT}-Tr=(2.87)^{1.3}-1.5>1$.

10) Here we are assuming that $\left(\beta'+\frac{T}{a+b-1}\beta\right)<0$. Note that $0<(a+b-1)<1$, and $\beta'>0, \beta<0$.

$$0 = a_{32}dC_0 - a_{33}dM_0 + b_3dr, \tag{37}$$

where $a = \frac{1-b}{r(1-a)}$, $a_{33}=1$, $b_3 = -\frac{(1-b)}{r^2(1-a)}$.

The equations (35), (36), and (37) can be written as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} dS_t/dr \\ dC_0/dr \\ dM_0/dr \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \text{ or } AX=y, \tag{38}$$

when $a_{11}=a_{21}=1$, $a_{12}=Q_1$, $a_{13}=Q_2$. Since $A = \begin{pmatrix} + & + & - \\ + & + & + \\ 0 & + & - \end{pmatrix}$ and $y = \begin{pmatrix} + \\ ? \\ + \end{pmatrix}$,[11)]

we have $|A|<0$

$$\frac{dS_T}{dr} = \frac{\begin{vmatrix} + & - & - \\ ? & + & + \\ + & + & + \end{vmatrix}}{(-)} = ?$$

$$\frac{dC_0}{dr} = \frac{\begin{vmatrix} + & + & - \\ + & ? & + \\ 0 & + & - \end{vmatrix}}{(-)} = ?$$

$$\frac{dM_0}{dr} = \frac{\begin{vmatrix} + & - & + \\ + & + & + \\ 0 & + & + \end{vmatrix}}{(-)} = ?$$

$$\frac{dM_0}{dr} = \frac{\begin{vmatrix} + & - & + \\ + & + & - \\ 0 & + & + \end{vmatrix}}{(-)} = \frac{(+)}{(-)} < 0.$$

Therefore we conclude that $\frac{dM_0}{dr}<0$ only if $b_2<0$, i.e., when the interest rate rises, *ceteris paribus*, the initial holdings of non-interest bearing assets will rise. Furthermore (33) implies that even if $\frac{dM_0}{dr}<0$, we cannot say for sure that $\frac{dM_t}{dr}<0$ for all t.

By the same method, we get the following system.

11) $b_2 = ZC_0^{\delta}M_0^{n}(\delta+n)T\gamma' e^{rT(\delta+n)} - M_0\gamma' Te^{rT} = (+)-(+)=?$

$$\begin{pmatrix} 1 & Q_1 & Q_2 \\ 1 & a_{22} & a_{23} \\ 0 & a_{32} & -1 \end{pmatrix} \begin{pmatrix} dS_T/dS_0 \\ dC_0/dS_0 \\ dM_0/dS_0 \end{pmatrix} = \begin{pmatrix} e^{rt} \\ 0 \\ 0 \end{pmatrix} \tag{39}$$

Thus we have

$$\frac{dS_T}{dS_0} = \frac{(-)}{(-)} > 0 \tag{40}$$

$$\frac{dC_0}{dS_0} = \frac{(+)}{(-)} < 0 \tag{41}$$

$$\frac{dM_0}{dS_0} = \frac{(+)}{(-)} < 0 \tag{42}$$

Again the results of (41) and (42) must not be interpreted as $\frac{dC_t}{dS_0} < 0$ for all t; it should be interpreted in terms of (32) and (33).

When the flow of income, w, is changed, *ceteris paribus*, we have the following results.

$$\begin{pmatrix} 1 & Q_1 & Q_2 \\ 1 & a_{22} & a_{23} \\ 0 & a_{32} & -1 \end{pmatrix} \begin{pmatrix} dS_T/dw \\ dC_0/dw \\ dM_0/dw \end{pmatrix} = \begin{pmatrix} \beta \\ 0 \\ 0 \end{pmatrix} \tag{43}$$

Thus $$\frac{dS_T}{dw} = \frac{(+)}{(-)} < 0 \tag{44}$$

$$\frac{dC_0}{dw} = \frac{(-)}{(-)} > 0 \tag{45}$$

$$\frac{dM_0}{dw} = \frac{(+)}{(-)} < 0. \tag{46}$$

From (32) and (45), it is clear that $\frac{dC_t}{dw} > 0$ for all t. It is not, however, certain whether $\frac{dM_t}{dw} > 0$ for all t; (46) implies that the initial holdings of M_0 will be smaller when the flow of income w rises but according to (33), it is not certain $\frac{dM_t}{dw} < 0$ for all t.

IV. As we have shown in (28)—(31), we can solve the time paths of C_t, M_t and S_t in terms of S_0, w, n and other parameters of the model. In general the solutions can be written as

$$C_t = C_t(S_0, w, r, t) \tag{47}$$

$$M_t = M_t(S_0, w, r, t) \tag{48}$$

$$S_t = S_t(S_0, w, r, t). \tag{49}$$

In most of the current monetary dynamic models, the signs of the partial derivatives of (47)—(49) are assumed away. In section Ⅲ, we have shown that the signs of those partial derivatives are not the same as they are commonly assumed except $\frac{dC_t}{dw}>0$.

Of course our investigation was based on a reasonable concave utility function and the assumption of certainty, i.e., the assumption that there is no expectation on r. The model can be extended to bring in the expectation and to generalize the comparative dynamic results for a general utility function.

REFERENCES

1. Atkinson, A. B., "Capital Taxes, the Redistribution of Wealth and Individual Savings," *Review of Economic Studies*, April, 1971, pp. 209-227.
2. Fisher, Stanley, "Keynes-Wicksell and Neo-classical Models of Money and Growth," *American Economic Review*, Dec., 1972, pp. 880—90.
3. Johnson, Harry, "Monetary Theory and Policy," *American Economic Review*, June, 1962.
4. Jorgenson, Dale W., "Discussion," *American Economic Review*, May, 1967, pp. 557—59.
5. Patinkin, Don, *Money, Interest, and Prices*, Row-Peterson, 1956.
6. Sidrauski, M., "Rational Choice and Patterns of Growth in A Monetary Economy," *American Economic Review*, May, 1967, pp. 534—44.
7. ________, "Inflation and Economic Growth," *Journal of Political Economy*, Dec., 1967, pp. 796—810.
8. Stein, J. L. and Nagatani, K., "Stabilization Policies in a Growing Economy," *Review of Economic Studies*, April, 1969, pp. 164—183.
9. Tobin, J., "Money and Economic Growth," *Econometrica*, Oct., 1965, pp. 671—684.