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ON CHAINS OF K-TRANSFORMS

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1. Introduction

If $g(y)$ and $f(x)$ are related by the integral equation

$$g(y)=\int_0^\infty f(x)k_\nu(xy)\sqrt{xy}\,dx, \tag{1.1}$$

then $g(y)$ is said to be the *K-transform* of order ν of $f(x)$ and regard y as a complex variable.

We shall denote (1.1) symbolically as

$$g(y)=M^\nu[f(x)] \tag{1.2}$$

This transform was introduced by C. S. Meijer [1]. The transform was further investigated by Boas [2], [3] and Erdélyi [4].

Recently Rathie [6] and Maheshwari [7] have studied the properties of the aforesaid transform by considering certain chains of this transform.

The object of this paper is to establish further some generalized results in transform (1.1) by using chains of this transform.

2. Theorem 1

If

$$M^\nu[f_1(x)]=g(y), \tag{2.1}$$

$$M^\nu[f_2(x)]=\pi f_1\left(\frac{1}{y}\right), \tag{2.2}$$

then

$$M^{2\nu}\left[x^{\frac{3}{2}}f_2\left(\frac{x^2}{4}\right)\right]=4y^{\frac{3}{2}}g(y^2), \tag{2.3}$$

provided $x^{\pm\nu+\frac{1}{2}}f_2(x)$ *are bounded and absolutely integrable in* $(0,\infty)$.

Further let

$$M^{2\nu}[f_3(x)]=\frac{\pi}{4}y^{-\frac{3}{2}}f_2\left(\frac{1}{4y^2}\right), \tag{2.4}$$

$$M^{2^2\nu}[f_4(x)]=\frac{\pi}{4}y^{-\frac{3}{2}}f_3\left(\frac{1}{4y^2}\right), \tag{2.5}$$

....

....

$$M^{2^{n-1}\nu}[f_n(x)]=\frac{\pi}{4}y^{-\frac{3}{2}}f_{n-1}\left(\frac{1}{4y^2}\right), \tag{2.6}$$

then

$$M^{2^{n-1}\nu}\left[x^{\frac{3}{2}}f_n\left(\frac{x^2}{4}\right)\right]=4y^{\frac{3}{2}(2^{n-1}-1)}g(y^{2(n-1)}), \tag{2.7}$$

provided $x^{\left(\pm 2^{n-2}\nu+\frac{1}{2}\right)}f_n(x)$, $n=2, 3, \cdots, n$, *are bounded and absolutely integrable in* $(0, \infty)$.

PROOF. Substituting the expression for $f_1(x)$ from (2.2) in (2.1), interchanging the order of integration which is justifiable under the conditions mentioned in the theorem, and evaluating the later integral by [5, p. 146], we obtain

$$g(y)=y^{-\frac{1}{2}}\int_0^\infty \sqrt{t}\, f_2(t)k_{2\nu}(2\sqrt{ty})dt. \tag{2.8}$$

Writing $y=y^2$ and $t=\frac{t^2}{4}$, we obtain from (2.8)

$$4y^{\frac{3}{2}}g(y^2)=\int_0^\infty t^{\frac{3}{2}}f_2\left(\frac{t^2}{4}\right)k_{2\nu}(ty)\sqrt{ty}\,dt$$

$$\text{i.e.}\quad M^{2\nu}\left[x^{\frac{3}{2}}f_2\left(\frac{x^2}{4}\right)\right]=4y^{\frac{3}{2}}g(y^2).$$

Proceeding successively we assume the result (2.7).

Also let

$$\frac{\pi}{4}y^{-\frac{3}{2}}f_n\left(\frac{1}{4y^2}\right)=\int_0^\infty f_{n+1}(x)k_{2(n-1)_\nu}(xy)\sqrt{xy}\,dx. \tag{2.9}$$

Substituting the expression for $f_n\left(\frac{x^2}{4}\right)$ from (2.9) in (2.7), interchanging the order of integration and evaluating the later integral by [5, p. 146], we obtain

$$y^{\frac{3}{2}(2^{n-1}-1)}g(y^{2^{(n-1)}})=\frac{1}{\sqrt{y}}\int_0^\infty \sqrt{t}\,f_{n+1}(t)k_{2^n\nu}(2\sqrt{ty})\,dt. \tag{2.10}$$

Writing $y=y^2$ and $t=\frac{t^2}{4}$, we obtain from (2.10)

$$4y^{\frac{3}{2}(2^n-1)}g(y^{2^n})=\int_0^\infty t^{\frac{3}{2}}f_{n+1}\left(\frac{t^2}{4}\right)k_{2^n\nu}(ty)\sqrt{ty}\,dt$$

$$\text{i.e.}\quad M^{2^n\nu}\left[x^{\frac{3}{2}}f_{n+1}\left(\frac{x^2}{4}\right)\right]=4y^{\frac{3}{2}(2^n-1)}g(y^{2^n}).$$

We thus find that if (2.7) is true for n, it is also true for $(n+1)$ i.e. for the next higher order. But we have seen that it is true for $n=2$ and so it is true for $n=3$ and so on. Hence (2.7) is true for all positive integral values of n except 1.

3. Theorem 2

If

$$M^{\nu}[f_1(x)]=g(y), \tag{3.1}$$

$$M^{\nu}[f_2(x)]=\pi y^{-2}f_1\left(\frac{1}{y}\right), \tag{3.2}$$

then

$$M^{2\nu}\left[x^{-\frac{1}{2}}f_2\left(\frac{x^2}{4}\right)\right]=y^{-\frac{1}{2}}g(y^2), \tag{3.3}$$

provided $x^{\left(\pm\nu\pm\frac{1}{2}\right)}f_2(x)$ *are bounded and absolutely integrable in* $(0, \infty)$.

Further if

$$M^{2\nu}[f_3(x)]=\pi y^{-\frac{3}{2}}f_2\left(\frac{1}{4y^2}\right), \tag{3.4}$$

$$M^{2^2\nu}[f_4(x)]=\pi y^{-\frac{3}{2}}f_3\left(\frac{1}{4y^2}\right), \tag{3.5}$$

....

....

$$M^{2^{n-2}\nu}[f_n(x)]=\pi y^{-\frac{3}{2}}f_{n-1}\left(\frac{1}{4y^2}\right), \tag{3.6}$$

then

$$M^{2^{n-1}\nu}\left[x^{-\frac{1}{2}}f_n\left(\frac{x^2}{4}\right)\right]=y^{-\left(2^{n-1}-\frac{1}{2}\right)}g(y^{2^{(n-1)}}), \tag{3.7}$$

provided $x^{\left(\pm 2^{n-2}\nu\pm\frac{1}{2}\right)}f_n(x)$, $n=2, 3, \cdots, n$, *are bounded and absolutely integrable in* $(0, \infty)$.

PROOF. In proving this theorem we make use of the well known result [5, p. 146]

$$\int_0^\infty x^{-\frac{5}{2}}k_\nu\left(\frac{a}{x}\right)k_\nu(xy)\sqrt{xy}\,dx=\frac{\pi}{a}\sqrt{y}\ \ k_{2\nu}(2\sqrt{ay})$$

Re $(a)>0$, Re $(y)>0$.

Proof of the theorem is omitted, as being similiar to that of the theorem 1.

4. Examples on the theorems

EXAMPLE 1. Let

$$f_1(x)=\sqrt{\pi}\,2^{-\nu}a^{(2\nu-1)}x^{2\nu}J_{\nu-\frac{1}{2}}\left(\frac{a^2x}{2}\right)$$

then making use of result [5, p. 137], we obtain from (2.1)

$$g(y)=\frac{\sqrt{\pi}\,a^{(4\nu-2)}}{y^{\left(3\nu+\frac{1}{2}\right)}}\Gamma(2\nu+1/2)\left(1+\frac{a^4}{4y^2}\right)^{-2\nu-1/2}$$

$$\mathrm{Re}(\nu)>-1/4,\quad \mathrm{Re}(y)>\left|\mathrm{Im}\frac{a^2}{2}\right|$$

From (2.2) and [5, p. 148], we obtain

$$f_2(x)=\frac{x^{\left(\nu-\frac{1}{2}\right)}}{\pi}I_{2\nu-1}(a\sqrt{x})\,J_{2\nu-1}(a\sqrt{x})$$

$$\mathrm{Re}(\nu)>0,\quad \mathrm{Re}(y)>0.$$

Taking $n=2$, we obtain from (2.7)

$$M^{2\nu}\left[\frac{x^{\left(2\nu+\frac{1}{2}\right)}}{2^{(2\nu-1)}\pi}I_{2\nu-1}(ax/2)J_{2\nu-1}(ax/2)\right]$$

$$=\frac{4\sqrt{\pi}\,a^{(4\nu-2)}}{y^{(6\nu-1/2)}}\Gamma(2\nu+1/2)\left(1+\frac{a^4}{4y^4}\right)^{-2\nu-\frac{1}{2}}$$

$$\mathrm{Re}(\nu)>0,\quad \mathrm{Re}(y)>\mathrm{Re}(a/2).$$

EXAMPLE 2.

Let

$$f_1(x)=\sqrt{\pi}\,2^{-(\nu+1)}a^{(2\nu+1)}x^{(2\nu+2)}J_{\nu-\frac{1}{2}}(a^2x/2),$$

then making use of the result [5, p. 137], we obtain

$$g(y)=\frac{2\sqrt{\pi}\,a^{4\nu}}{\Gamma\left(\frac{1}{2}+\nu\right)}y^{-\left(3\nu+\frac{5}{2}\right)}\Gamma(2\nu+3/2)\Gamma(\nu+3/2)\,{}_2F_1\left(2\nu+3/2,\nu+3/2\,;\;1/2+\nu\,;\;-\frac{a^4}{4y^2}\right)$$

$$\mathrm{Re}(\nu)>-3/4,\quad \mathrm{Re}(y)>\left|\mathrm{Im}\,\frac{a^2}{2}\right|.$$

From (2.2) and [5, p. 148], we obtain

$$f_2(x)=\frac{x^{\left(\nu+\frac{1}{2}\right)}}{\pi}I_{2\nu}(a\sqrt{x})J_{2\nu}(a\sqrt{x}),$$

$$\mathrm{Re}(\nu)>-1/2,\quad \mathrm{Re}(y)>0.$$

Taking $n=2$, we obtain from (2.7)

$$M^{2\nu}\left[\frac{x^{\left(2\nu+\frac{5}{2}\right)}}{2^{(2\nu+1)}\pi}I_{2\nu}(ax/2)J_{2\nu}(ax/2)\right]$$

$$=\frac{8\sqrt{\pi}\,a^{4\nu}}{\Gamma\left(\frac{1}{2}+\nu\right)}y^{-\left(6\nu+\frac{7}{2}\right)}\Gamma(2\nu+3/2)\Gamma(\nu+3/2)\,{}_2F_1\left(2\nu+3/2,\ \nu+3/2\,;\;1/2+\nu\,;\;-\frac{a^4}{4y^4}\right)$$

$$\mathrm{Re}(\nu)>-1/2,\ \mathrm{Re}(y)>\mathrm{Re}(a/2).$$

EXAMPLE 3.

Let

$$f_1(x)=\sqrt{\pi}\,2^{-\nu}a^{(2\nu-1)}x^{(2\nu-2)}J_{\nu-\frac{1}{2}}(a^2x/2),$$

then making use of the result [5, p. 137], we obtain from (3.1)

$$g(y)=\frac{\sqrt{\pi}\,a^{(4\nu-2)}}{4\Gamma\left(\nu+\frac{1}{2}\right)y^{\left(3\nu-\frac{3}{2}\right)}}\Gamma(2\nu-1/2)\Gamma(\nu-1/2)$$

$$\times{}_2F_1\left(2\nu-1/2,\ \nu-1/2\,;\,\nu+1/2\,;\,-\frac{a^4}{4y^2}\right),$$

$$\mathrm{Re}(\nu)>\frac{1}{2},\ \mathrm{Re}(y)>\left|\mathrm{Im}\left(\frac{a^2}{2}\right)\right|.$$

From (3.2) and [5, p. 148], we obtain

$$f_2(x)=\frac{x^{\left(\nu-\frac{1}{2}\right)}}{\pi}I_{2\nu-1}(a\sqrt{x})J_{2\nu-1}(a\sqrt{x})$$

$$\mathrm{Re}(\nu)>0,\ \mathrm{Re}(y)>0.$$

Taking $n=2$ we obtain from (3.7)

$$M^{2\nu}\left[\frac{x^{\left(2\nu-\frac{3}{2}\right)}}{2^{(2\nu-1)}\pi}I_{2\nu-1}(ax/2)J_{2\nu-1}(ax/2)\right]=\frac{\sqrt{\pi}\,a^{(4\nu-2)}}{4\Gamma\left(\nu+\frac{1}{2}\right)y^{\left(3\nu-\frac{3}{2}\right)}}\Gamma(2\nu-1/2)$$

$$\times(\nu-1/2)\,{}_2F_1\left(2\nu-1/2,\ \nu-1/2\,;\,\nu+1/2\,;\,-\frac{a^4}{4y^4}\right)$$

$$\mathrm{Re}(\nu)>1/2,\ \mathrm{Re}(y)>\mathrm{Re}(a/2).$$

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REFERENCES

[1] Meijer, C. S., (1940), Proc. Amsterdam Akad. Wet. 43, 599—608 and 702—711.
[2] Boas, R. P., (1942a), Proc. Nat. Acad. Sci. U. S. A. 28, 21—24.
[3] ________, (1942b), Bull. Amer. Math. Soc. 48, 286—294.
[4] Erdélyi, Arthur, (1950—51), Rend. Sem. Mat. Univ. Torino 10, 217—234.
[5] The Bateman Manuscripts Project, Tables of integral Transforms, Vol. 2, New York, McGraw-Hill Book Co., 1954.
[6] Rathie, P. N., (1964), Annales de la Société Scientifique de Bruxelles, T. 78, III, pp. 171—174.
[7] Maheshwari, M. L., (1967), Thesis for the Ph. D. degree of the B. I. T. S. Pilani (Rajasthan), India.