Kyungpook Math. J.
Volume 12, Number 2
December, 1972

SOME DOUBLE INTEGRAL TRANSFORMATION OF MEIJER'S *G*-FUNCTION

By N. C. Jain and Rajendra K. Saxena*

1. Introduction. Rainville [5, p. 104], Abdul Halim and Al-Salam [1] have shown that the single and double Euler transformations of the hypergeometric function ${}_pF_q$ are effective tools for augmenting its parameters. Recently Shrivastava and Singhal [8] and Shrivastava and Joshi [9] have discussed some similar interesting properties of ${}_pF_q$ in double Meijer and double Whittaker transforms respectively.

The object of the present paper is to establish a double integral transform of Meijer's G-function which leads to yet another interesting process of augmenting the parameters in the G-function. The result is of general character and on specialising the parameters suitably, yields several interesting results as particular cases.

In what follows for the sake of brevity we have used the symbols (a_r), $\triangle(r, a)$, $\triangle(r, \pm a)$, $\triangle((r, a_p))$ to denote the set of parameters $a_1, a_2, \cdots\cdots, a_r$; $\frac{a}{r}, \frac{a+1}{r}, \cdots\cdots, \frac{a+r-1}{r}$; $\triangle(r, a)$, $\triangle(r, -a)$ and $\triangle(r, a_1)$, $\triangle(r, a_2)$, $\cdots\cdots$, $\triangle(r, a_p)$ respectively.

2. In this section, we have established the following double integral transform of G-function.

If s, k and r are positive integers, then

$$(2.1)\quad \int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}(x+y)^\sigma\, G^{f,g}_{u,v}\left(\lambda(x+y)\Big|{}^{(c_u)}_{(d_v)}\right) G^{m,n}_{p,q}\left(tx^s y^k(x+y)^r\Big|{}^{(a_p)}_{(b_q)}\right)dx\,dy$$

$$=(2\pi)^{(1-D)\left(f+g-\frac{1}{2}u-\frac{1}{2}v\right)+\frac{1}{2}}D^{\sum_1^v d_j-\sum_1^u c_j+\left(A-\frac{1}{2}\right)(v-u)}\,\frac{s^{\alpha-\frac{1}{2}}k^{\beta-\frac{1}{2}}}{\lambda^{\alpha+\beta+\sigma}(s+k)^{\alpha+\beta-\frac{1}{2}}}\times$$

---

* formerly R. K. Saxena

$$\times G^{m+Dg,\, n+\rho+Df}_{p+\rho+Dv,\, q+p+Du}\left(\frac{t\delta D^{D(v-u)}}{\lambda^D}\,\middle|\,\begin{matrix}\triangle((D,1-A-d_f)) & \triangle(s,1-\alpha),\triangle(k,1-\beta),(a_p), \\ \triangle((D,1-A-C_g)), & (b_q),\triangle(k+s,1-\alpha-\beta),\end{matrix}\right.$$
$$\left.\begin{matrix}\triangle(D,1-A-d_{f+1}),\cdots,\triangle(D,1-A-d_v) \\ \triangle(D,1-A-C_{g+1}),\cdots,\triangle(D,1-A-C_u)\end{matrix}\right),$$

where $\delta=\dfrac{s^s k^k}{(s+k)^{s+k}},\ \rho=s+k,\ D=s+k+r,\ A=\sigma+\alpha+\beta$:

$0\le Dg\le Du\le Dv<Du+q-p,\ u+v-2g\le 2f\le 2v,\ 0\le n\le p,\ p+q-2n<2m\le 2q;$

$\mathrm{Re}(\min_i d_i+D\min_j b_j)>\mathrm{Re}(-A)>\mathrm{Re}\left[D\left(\frac{s-\alpha}{s},\ \frac{k-\beta}{k},\ a_l\right)+C_t-D-1\right],\ i=1,2,\cdots\cdots,$ $f;\ j=1,2,\cdots\cdots,\ m;\ l=1,2,\cdots\cdots,\ n;\ t=1,\ 2,\cdots\cdots,\ g;\ u.$ $\mathrm{Re}(\min C_i+A)-v.$ $\mathrm{Re}(\max d_j+A)-uD+v+\frac{1}{2}D(Dv-Du+1)>D(Dv-Du).$ $\mathrm{Re}\max\left(\frac{s-\alpha}{s},\ \frac{k-\beta}{k},\ a_l\right),$ $i=1,2,\cdots\cdots,\ u;\ j=1,2,\cdots\cdots,v\ ;\ l=1,2,\cdots\cdots,\ u;\ |\arg\lambda|\le\left(f+g-\frac{1}{2}u-\frac{1}{2}v\right)\pi,$ $|\arg t|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi,\ \mathrm{Re}(\alpha+sb_j)>0,\ \mathrm{Re}(\beta+kb_j)>0,\ j=1,2,\cdots\cdots,\ m,$ *and the double integral converges.*

PROOF. To prove (2,1), we start with the following known result [2, p. 177]

$$(2.2)\qquad \int_0^\infty\int_0^\infty \phi(x+y)x^{\alpha-1}y^{\beta-1}dx\,dy=B(\alpha,\beta)\int_0^\infty\phi(z)z^{\alpha+\beta-1}dz$$

which is valid for $\mathrm{Re}(\alpha)>0$ and $\mathrm{Re}(\beta)>0$.

It is easy to prove by following the technique of reversing the order of integrations, that

$$(2.3)\qquad \int_0^\infty\int_0^\infty \phi(x+y)x^{\alpha-1}y^{\beta-1}G^{m,n}_{p,q}\left(tx^s y^k(x+y)^r\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right)dx\,dy$$
$$=\sqrt{2\pi}\frac{s^{\alpha-\frac{1}{2}}k^{\beta-\frac{1}{2}}}{(s+k)^{\alpha+\beta-\frac{1}{2}}}\int_0^\infty\phi(z)z^{\alpha+\beta-1}G^{m,n+\rho}_{p+\rho,\,q+\rho}\left(t\delta z^D\,\middle|\,\begin{matrix}\triangle(s,1-\alpha),\triangle(k,1-\beta),(a_p)\\(b_q),\triangle(k+s,1-\alpha-\beta)\end{matrix}\right)dz$$

where s,k and r are positive integers,

$\delta=\dfrac{s^s k^k}{(s+k)^{s+k}},\ \rho=s+k,\ D=s+k+r,\ p+q<2(m+n),\ |\arg t|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi,$

$\mathrm{Re}(\alpha+sb_j)>0,\ \mathrm{Re}\ (\beta+kb_j)>0,\ j=1,2,\cdots\cdots,\ m$:

In (2.3), taking

$$\phi(z)=z^\sigma G^{f,g}_{u,v}\left(\lambda z\,\middle|\,\begin{matrix}(c_u)\\(d_v)\end{matrix}\right)$$

and evaluating the integral on the right hand side using [7, *p.* 401] the result (2.1) follows.

3. Particular cases. On choosing the parameters suitably in (2.1), several known and unknown results are obtained as particular cases. However, we mention some of the interesting results here.

(a) Taking $f=v=2,\ g=0,\ u=1,\ c_1=\frac{1}{2},\ d_1=\nu,\ d_2=-\nu,\ \sigma=\mu+\frac{1}{2}$ in (2.1) and using [3, p. 216, (5)]

$$G_{1,2}^{2,0}\left(x\,\middle|\,{\textstyle\frac{1}{2} \atop b,-b}\right)=\pi^{-\frac{1}{2}}e^{-\frac{1}{2}x}K_b\left(\tfrac{1}{2}x\right),$$

we obtain

$$(3.1)\quad \int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}(x+y)^{\mu+\frac{1}{2}}e^{-\frac{1}{2}\lambda(x+y)}K_\nu\left\{\tfrac{1}{2}\lambda(x+y)\right\}\times$$

$$\times G_{p,q}^{m,n}\left(tx^sy^k(x+y)^r\,\middle|\,{(a_p) \atop (b_q)}\right)dxdy$$

$$=(2\pi)^{\frac{1}{2}(2-D)}D^{A-1}\sqrt{\pi}\,\frac{s^{\alpha-\frac{1}{2}}k^{\beta-\frac{1}{2}}}{\lambda^{\sigma+\alpha+\beta}(s+k)^{\alpha+\beta-\frac{1}{2}}}\times$$

$$\times G_{p+\rho+2D,\,q+\rho+D}^{m,\,n+\rho+2D}\left(t\delta\left(\frac{D}{\lambda}\right)^D\,\middle|\,{\triangle(D,1-A\mp\nu),\ \triangle(s,1-\alpha),\ \triangle(k,1-\beta),\ (a_p) \atop (b_q),\ \triangle\left(D,\frac{1}{2}-A\right),\ \triangle(k+s,\ 1-\alpha-\beta)}\right),$$

where δ, D and λ have the same value as (2.1) and $A=\mu+\alpha+\beta+\frac{1}{2}$;

$p+q<2(m+n)$, $\mathrm{Re}(\alpha+sb_j\pm\nu)>0$, $\mathrm{Re}(\beta+kb_j\pm\nu)>0$, $\mathrm{Re}\left(\alpha+\beta+\mu\pm\nu+Db_j+\frac{1}{2}\right)>0$, $j=1,2,\cdots\cdots,m$, $\mathrm{Re}(\lambda)>0$, $|\arg t|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi$.

(b) Further, replacing q, t and (a_p) by $q+1$, $-t$ and $(1-a_p)$ respectively and then putting $m=1$, $n=p$, $b_1=0$, $b_{j+1}=1-b_j$ $(j=1,2,\cdots,q)$, using the result [3, p. 215, (1)] and [3, p. 4, (11)] we obtain an interesting result.

$$(3.2)\quad \int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}(x+y)^{\mu+\frac{1}{2}}e^{-\frac{1}{2}\lambda(x+y)}\ K_\nu\left\{\tfrac{1}{2}\lambda(x+y)\right\}{}_pF_q\left[{(a_p) \atop (b_q)}:tx^sy^k(x+y)^r\right]dxdy$$

$$=\frac{\sqrt{\pi}\,\Gamma\left(\alpha+\beta+\mu\pm\nu+\frac{1}{2}\right)}{\lambda^{\alpha+\beta+\mu+\frac{1}{2}}\Gamma(\alpha+\beta+\mu+1)}B(\alpha,\beta)\times{}_{p+3s+3k+2r}F_{q+2s+2k+r}\left[{(a_p), \atop (b_q),}\right.$$

$$\left.\begin{matrix}\triangle(s,\alpha),\ \triangle(k,\beta),\ \triangle\left(s+k+r,\ \alpha+\beta+\mu\pm\nu+\frac{1}{2}\right)\\ \triangle(s+k,\ \alpha+\beta),\ \triangle(s+k,\ \alpha+\beta+\mu+1)\end{matrix} : t\delta\left(\frac{s+k+r}{\lambda}\right)^{s+k+r}\right],$$

provided $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, $\mathrm{Re}(\lambda)>0$, $\mathrm{Re}\left(\alpha+\beta+\mu\pm\nu+\frac{1}{2}\right)>0$, recently obtained by Shrivastava and Singhal [8].

(c) Setting $v=f=2$, $g=0$, $u=1$, $c_1=1-\mu$, $d_1=\frac{1}{2}+\nu$, $d_2=\frac{1}{2}-\nu$ in (2. 1) and using the known formula [3, p. 216, (6)]

$$G_{1,2}^{2,0}\left(x\ \middle|\ \begin{matrix}1-k\\ \frac{1}{2}+m,\frac{1}{2}-m\end{matrix}\right)=e^{-\frac{1}{2}x}W_{k,m}(x),$$

we have

$$(3.3)\quad \int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}(x+y)^\sigma e^{-\frac{1}{2}\lambda(x+y)}W_{\mu,\nu}[\lambda(x+y)]G_{p,q}^{m,n}\left(tx^sy^k(x+y)^r\ \middle|\ \begin{matrix}(a_p)\\(b_q)\end{matrix}\right)dxdy$$

$$=(2\pi)^{\frac{1}{2}(2-D)}D^{\mu+A-\frac{1}{2}}\frac{s^{\alpha-\frac{1}{2}}k^{\beta-\frac{1}{2}}}{\lambda^{\alpha+\beta+\sigma}(s+k)^{\alpha+\beta-\frac{1}{2}}}\times$$

$$\times G_{p+\rho+2D,q+\rho+D}^{m,n+\rho+2D}\left(t\delta\left(\frac{D}{\lambda}\right)^D\ \middle|\ \begin{matrix}\triangle\left(D,\frac{1}{2}-A\pm\nu\right),\triangle(s,1-\alpha),\triangle(k,1-\beta),(a_p)\\ (b_q),\triangle(k+s,1-\alpha-\beta),\triangle(D,\mu-A)\end{matrix}\right),$$

where D,ρ,δ and A are given in (2.1) ;

$p+q<2(m+n)$, $|\arg t|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi$, $\mathrm{Re}(\lambda)>0$, $\mathrm{Re}(\alpha+sb_j)>0$, $\mathrm{Re}(\beta+kb_j)>0$, $\mathrm{Re}\left(\alpha+\beta+\sigma+Db_j\pm\nu+\frac{1}{2}\right)>0$, $j=1,2,\cdots,m$.

(d) Further, replacing q, t and (a_p) by $q+1$, $-t$ and $(1-a_p)$ respectively and then putting $m=1$, $n=p$, $b_1=0$, $b_{j+1}=1-b_j$ $(j=1,2,\cdots,q)$ and using the result [3, p. 215, (1)], (3. 3) reduces to a result due to Shrivastava and Joshi [9, p. 19, (2. 3)]

$$(3.4)\quad \int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}(x+y)^\sigma e^{-\frac{1}{2}\lambda(x+y)}W_{\mu,\nu}[\lambda(x+y)]\,{}_pF_q\left[\begin{matrix}(a_p)\\(b_q)\end{matrix}: tx^sy^k(x+y)^r\right]dxdy$$

$$=\frac{\Gamma\left(\alpha+\beta+\sigma\pm\nu+\frac{1}{2}\right)}{\lambda^{\alpha+\beta+\sigma}\Gamma(\alpha+\beta+\sigma-\mu+1)}B(\alpha,\beta)\times$$

$$\times{}_{p+3\rho+2r}F_{q+2\rho+r}\left[\begin{matrix}(a_p),\triangle(s,\alpha),\triangle(k,\beta),\ \triangle\left(\rho+r,\alpha+\beta+\sigma\pm\nu+\frac{1}{2}\right)\\ (b_q),\triangle(\rho,\alpha+\beta),\triangle(\rho+r,\alpha+\beta+\sigma-\mu+1)\end{matrix};\ t\delta\delta'\right]$$

where

$$\rho=s+k,\ \delta=\frac{s^sk^k}{(s+k)^{s+k}},\ \delta'=\left(\frac{s+k+r}{\lambda}\right)^{s+k+r}$$

$\operatorname{Re}(\alpha)>0$, $\operatorname{Re}(\beta)>0$, $\operatorname{Re}(\lambda)>0$, $\operatorname{Re}\left(\alpha+\beta+\sigma\pm\nu+\frac{1}{2}\right)>0$ and the resulting hypergeometric series converges.

With $\mu=0$, $\nu=\pm\frac{1}{2}$ and $\sigma=-\frac{1}{2}$, (3.4) will reduce to the earlier results of Jain [4] and Singh [6].

(e) Choosing $f=g=u=1$, $v=2$, $c_1=1-k$, $d_1=\frac{1}{2}+M$, $d_2=\frac{1}{2}-M$ in (2.1) and using the known result

$$G_{1,2}^{1,1}\left(x\,\middle|\,\begin{matrix}1-k\\ \frac{1}{2}+m,\frac{1}{2}-m\end{matrix}\right)=\frac{\Gamma\left(\frac{1}{2}+k+m\right)}{\Gamma(2m+1)}e^{-\frac{1}{2}x}M_{k,m}(x)$$

we obtain

$$(3.5)\quad \int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}(x+y)^\sigma e^{-\frac{1}{2}\lambda(x+y)}M_{k,M}\,[\lambda(x+y)]\,G_{p,q}^{m,n}\left(tx^sy^k(x+y)^r\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right)dxdy$$

$$=(2\pi)^{\frac{1}{2}(2-D)}D^{k+A-\frac{1}{2}}\frac{\Gamma(2M+1)}{\Gamma\left(k+M+\frac{1}{2}\right)}\frac{s^{\alpha-\frac{1}{2}}k^{\beta-\frac{1}{2}}}{\lambda^{\alpha+\beta+\sigma}(s+k)^{\alpha+\beta-\frac{1}{2}}}\times$$

$$\times G_{p+\rho+2D,q+\rho+D}^{m+D,n+\rho+D}\left(t\delta\left(\frac{D}{\lambda}\right)^D\,\middle|\,\begin{matrix}\triangle\left(D,\frac{1}{2}-A-m\right),\triangle(s,1-\alpha),\triangle(k,1-\beta),(a_p),\triangle\left(D,\frac{1}{2}-A+M\right)\\ \triangle(D,k-A),(b_q),\triangle(s+k,1-\alpha-\beta)\end{matrix}\right),$$

where ρ, δ, D and A have the same value as in (2.1);

$p+q<2(m+n)$, $|\arg t|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi$, $\operatorname{Re}(\lambda)>0$, $\operatorname{Re}(\alpha+sb_j)>0$,

$\operatorname{Re}(\beta+kb_j)>0$, $\operatorname{Re}\left(\alpha+\beta+\sigma+Db_j+M+\frac{1}{2}\right)>0$, $j=1,2,\cdots,m$.

(f) Substituting $f=1$, $g=u=0$, $v=2$, $d_1=\frac{1}{2}\nu$, $d_2=-\frac{1}{2}$ and using the result [3, p. 216, (3)]

$$G_{0,2}^{1,0}\left(x\,\middle|\,\begin{matrix}-\\ \frac{1}{2}\nu,-\frac{1}{2}\nu\end{matrix}\right)=J_\nu(2\sqrt{x}),$$

(2.1) reduces to

$$(3.6)\quad \int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}(x+y)^\sigma J_\nu(2\sqrt{\lambda(x+y)})\,G_{p,q}^{m,n}\left(tx^sy^k(x+y)^r\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right)dxdy$$

$$=\sqrt{2\pi}\frac{D^{2A-1}s^{\alpha-\frac{1}{2}}k^{\beta-\frac{1}{2}}}{\lambda^{\sigma+\alpha+\beta}(s+k)^{\alpha+\beta-\frac{1}{2}}}\times$$

$$\times G_{p+\rho+2D,q+\rho}^{m,n+\rho+D}\left(t\delta\left(\frac{D}{\lambda}\right)^D\,\middle|\,\begin{matrix}\triangle\left(D,1-A-\frac{1}{2}\nu\right),\ \triangle(s,1-\alpha),\triangle(k,1-\beta),\ (a_p),\triangle\left(D,1-A+\frac{1}{2}\nu\right)\\ (b_q),\triangle(k+s,1-\alpha-\beta)\end{matrix}\right)$$

where δ, D, ρ and A have the value given in (2.1) ;

$p+q<2(m+n)$, $|\arg t| < \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi$, $\mathrm{Re}(\lambda)>0$, $\mathrm{Re}(\alpha+sb_j)>0$, $\mathrm{Re}(\beta+kb_j)>0$, $\mathrm{Re}\left(\alpha+\beta+\sigma+\frac{1}{2}\nu+Db_j\right)>0$, $j=1,2,\cdots,m$, $\mathrm{Re}(\alpha+\beta+\sigma-D+Da_i)<1/4$, $i=1,2,\cdots,n$.

In view of the numerous properties of Meijer's G-function [3, p. 216—219], on specialising the parameters suitably, a large number of interesting results may be obtained as particular case.

Shri G. S. Institute of
Technology and Science,
Indore (India)

REFERENCES

[1] N. Abdul-Halim and W. A. Al-Salam, *Double Euler Transformation of certain Hypergeometric functions*, Duke Math. J., 30(1963), 51—62

[2] J. Edwards, *A Treatise on integral calculus*, Chelsea, New York, 1954.

[3] A. Erdelyi Et. Al., *Higher Transcendental functions*, Vol. I McGraw Hill, New York, 1953.

[4] R. N. Jain, *Some double integral transformations of certain Hypergeometric functions*, Math. Japon., 10, (1965), pp. 17—26.

[5] E. D. Rainville, *Special Functions*, Macmillan, New York

[6] R. P. Singh, *A note on double transformation of certain Hypergeometric functions*, Proc. Edinburgh. Math. Soc. (2), 14(1965), pp. 221—227.

[7] R. K. Saxena, *Some theorems on generalised Laplace transforms*, Proc. Nat. Inst. Sc. India, 26A (1960), pp. 400—413.

[8] H. M. Shrivastava and J. P. Singhal, *Double Meijer transformation of certain hypergeometric functions*, Proc. Camb. Phil. Soc. 64(1968).

[9] H. M. Shrivastava and C. M. Joshi, *Certain Double Whittaker transforms of generalised Hypergeometric functions*, The Yokohama Mathematical Journal, Vol. XV. (1)(1967), pp. 17—32.