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ON TRIGONOMETRICAL SERIES OF KAMPE' de FERIET'S DOUBLE HYPERGEOMETRIC FUNCTION OF HIGHER ORDER

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§ 1. Introduction

Kampe' de Feriet's [1] introduced the double hypergeometric function of higher order (i.e. with more parameters) in two variables, namely

$$(1.1)\qquad F\left(\begin{array}{c|c|c} \lambda & \alpha_1,\cdots\cdots\cdots,\alpha_\lambda & x \\ \mu & \beta_1,\beta'_1;\cdots\cdots;\beta_\mu,\beta'_\mu & \\ \nu & \gamma_1,\cdots\cdots\cdots,\gamma_\nu & \\ \sigma & \delta_1,\delta'_1;\cdots\cdots;\delta_\sigma,\delta'_\sigma & y \end{array}\right)=F\left(\begin{array}{c|c|c} \lambda & \alpha_\lambda & x \\ \mu & \beta_\mu,\beta'_\mu & \\ \nu & \gamma_\nu & \\ \sigma & \delta_\sigma,\delta'_\sigma & y \end{array}\right)$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{\lambda}(\alpha_j;\ m+n)\prod_{j=1}^{\mu}\{(\beta_j;m)(\beta'_j;n)\}}{\prod_{j=1}^{\nu}(\gamma_\nu;m+n)\prod_{j=1}^{\sigma}\{(\delta_j;m)(\delta'_j;n)\}}\cdot\frac{x^m y^n}{(l;m)(l;n)};$$

where $\lambda+\mu\leq\nu+\sigma+1$.

For the definition and properties of this function the reader is referred to [1], pp. 147—176. For special values of the parameters $\lambda,\mu,\nu,\sigma,$ the function (1.1) reduces to the double hypergeometric functions of Appel and the generalized hypergeometric functions.

The main result will be stated and proved in § 2; while some interesting particular cases will be deduced in § 3 and § 4. It may be noted that the constants and the parameters are such that the functions involved exist.

§ 2. Trigonometrical Series

The trigonometrical expansion to be established is

$$(2.1)\qquad \sum_{r=0}^{\infty}\frac{(-1)^r}{n!}\ \frac{\Gamma(n+r+1)\Gamma(S+1)\Gamma(n+2S+2)}{\Gamma(S-r+1)\Gamma(r+1)\Gamma(n+r+2+S)}$$

$$F\left(\begin{array}{l|l|c} \lambda+q+3 & \alpha_\lambda,\ 1+\frac{n}{2}+S,\ \frac{3}{2}+\frac{n}{2}+S,\ b_q+S,\ 1+S & x \\ \mu & \beta_\mu,\ \beta'_\mu & \\ \nu+p+3 & p_\nu, 1-r+S,\ a_p+S,\ n+r+2+S & \\ \sigma & \delta_\sigma,\delta'_\sigma & y \end{array}\right)\sin(n+2r+1)\theta$$

$$=\frac{\sqrt{\pi}}{2}\sum_{K=0}^{n}\frac{(\sin\theta)^{2S+1}(\cos\theta-1)^{K}}{K!(n-K)!}\cdot\frac{\Gamma(2+2S+n+K)}{2^{K}\Gamma\left(\frac{3}{2}+K+S\right)}$$

$$F\left(\begin{array}{l|l|l} \lambda+q+2 & \alpha_{\lambda}, \frac{2+n+K}{2}+S, \frac{3}{2}+\frac{n+K}{2}+S, b_q+S & x\sin^2\theta \\ \mu & \beta_{\mu}, \beta'_{\mu} & \\ \nu+p+2 & \rho_{\nu}, a_p+S, 1+S, \frac{3}{2}+K+S & y\sin^2\theta \\ \sigma & \delta_{\sigma}, \delta'_{\sigma} & \end{array}\right)$$

where $\lambda+\mu+q\leq\nu+\sigma+p+1,\quad 0\leq\theta\leq\pi$

PROOF. Since [(2), p. 135]

$$(2.2)\qquad \sum_{r=0}^{\infty}\frac{(n+r)!}{n!r!}G_{p+3,q+3}^{v+2,u+1}\left(Z\,\middle|\,\begin{array}{l} 1-r, a_1,\cdots,a_p, 1, n+r+2 \\ 1+\frac{n}{2}, \frac{3+n}{2}, b_1,\cdots,b_q, 1 \end{array}\right)\sin(n+2r+1)\theta$$

$$=\frac{\sqrt{\pi}}{2}\sum_{K=0}^{n}\frac{\sin\theta(\cos\theta-1)^{K}}{K!(n-K)!}G_{p+2,q+2}^{v+2,u}\left(\frac{Z}{\sin^2\theta}\,\middle|\,\begin{array}{l} a_p, 1, \frac{3}{2}+K \\ 1+\frac{n+K}{2}, \frac{3}{2}+\frac{n+K}{2}, b_q \end{array}\right),$$

where $0\leq\theta\leq\pi,\quad |\arg Z|<\left(v+u-\frac{1}{2}p-\frac{1}{2}q\right)\pi$, we have

$$(2.3)\quad \sum_{r=0}^{\infty}\frac{(n+r)!}{n!r!}\sin(n+2r+1)\theta\int_{0}^{\infty}Z^{s-1}f(Z)G_{p+3,q+3}^{v+2,u+1}\left(Z\,\middle|\,\begin{array}{l} 1-r, a_p, 1, n+r+2 \\ 1+\frac{n}{2}, \frac{3+n}{2}, b_q, 1 \end{array}\right)dZ$$

$$=\frac{\sqrt{\pi}}{2}\sum_{r=0}^{n}\frac{\sin\theta(\cos\theta-1)^{K}}{K!(n-K)!}\cdot$$

$$\int_{0}^{\infty}Z^{S-1}f(Z)G_{p+2,q+2}^{v+2,u}\left(\frac{Z}{\sin^2\theta}\,\middle|\,\begin{array}{l} a_p, 1, \frac{3}{2}+K \\ 1+\frac{n+K}{2}, \frac{3}{2}+\frac{n+K}{2}, b_q \end{array}\right)dZ,$$

provided the integrals involved exist. Now if we take

$$f(Z)=F\left(\begin{array}{l|l|l} \lambda & \alpha_1,\cdots\cdots,\alpha_{\lambda} & \\ \mu & \beta_1,\beta'_1;\cdots\cdots;\beta_{\mu},\beta'_{\mu} & xZ \\ \nu & \rho_1,\rho'_1;\cdots\cdots;\rho_{\nu},\rho'_{\nu} & yZ \\ \sigma & \delta_1,\delta'_1;\cdots\cdots;\delta_{\sigma},\delta'_{\sigma} & \end{array}\right)$$

in (2.3) and evaluate the integrals their in to get the desired result (2.1). The final form (2.1) comes after some simplifications.

§ 3. Fourier Series

By taking $n=0$, we obtain a Fourier sine series for Kampe'de Feriets function:

(3.1)
$$\sum_{r=0}^{\infty}(-1)^r \frac{1}{\Gamma(S-r+1)(S+r+2)}\cdot F\left(\begin{array}{c|l|c} \lambda+q+3 & \alpha_\lambda, 1+S, \frac{3}{2}+S, b_q+S, 1+S & x \\ \mu & \beta_\mu, \beta'_\mu & \\ \nu+p+3 & \rho_\nu, 1-r+S, a_p+S, 2+r+S & y \\ \sigma & \delta_\sigma, \delta'_\sigma & \end{array}\right)\cdot\sin(2r+1)\theta$$
$$=\frac{\sqrt{\pi}}{2}\frac{1}{\Gamma(S+1)\Gamma\left(S+\frac{3}{2}\right)}\cdot(\sin\theta)^{2S+1} \cdot F\left(\begin{array}{c|l|c} \lambda+q+2 & \alpha_\lambda, 1+S, \frac{3}{2}+S, b_q+S & x\sin^2\theta \\ \mu & \beta_\mu, \beta'_\mu & \\ \nu+p+2 & \rho_\nu, a_p+S, 1+S, \frac{3}{2}+S & y\sin^2\theta, \\ \sigma & \delta_\sigma, \delta'_\sigma & \end{array}\right)$$

where $0\leq\theta\leq\pi$, $\lambda+\mu+q\leq\nu+\sigma+p+1$.

§ 4. Particular Cases

The following particular cases are deduced by taking suitable values to parameters in (2.1) and (3.1)

(4.1)
$$\sum_{r=0}^{\infty}\frac{(-1)^r}{n!}\frac{\Gamma(n+r+1)\Gamma(S+1)\Gamma(n+2S+2)}{\Gamma(S-r+1)\Gamma(r+1)\Gamma(n+r+S+2)}\cdot$$
$$\cdot{}_{\lambda+q+5}F_{\nu+p+6}\left(\begin{array}{l} \alpha_\lambda, 1+\frac{n}{2}+S, \frac{3}{2}+\frac{n}{2}+S, b_q+S, 1+S, \frac{\delta_1+\delta'_1-1}{2}, \frac{\delta_1+\delta'_1}{2}; \\ \rho_\nu, 1-r+S, a_p+S, 2+n+r+S, \delta_1, \delta'_1, \delta_1+\delta'_1-1; \end{array} 4x\right)$$
$$\cdot\sin(n+2r+1)\theta$$
$$=\frac{\sqrt{\pi}}{2}\sum_{K=0}^{n}\frac{(\sin\theta)^{2S+1}(\cos\theta-1)^K}{K!(n-K)!}\cdot\frac{\Gamma(2+2S+n+K)}{2^K\Gamma\left(\frac{3}{2}+K+S\right)}$$
$$\cdot{}_{\lambda+q+4}F_{\nu+p+5}\left(\begin{array}{l} \alpha_\lambda, \frac{2+n+K}{2}+S, \frac{3}{2}+\frac{n+K}{2}+S, b_q+S, \frac{\delta_1+\delta'_1-1}{2}, \frac{\delta_1+\delta'_1}{2} \\ \rho_\nu, a_p+S, 1+S, \frac{3}{2}+K+S, \delta_1, \delta'_1, \delta_1+\delta'_1-1; \end{array} 4x\sin^2\theta\right)$$

where $\lambda+q\leq\nu+p+2$ and $0\leq\theta\leq\pi$.

(4.2)
$$\sum_{r=0}^{\infty}\frac{(-1)^r}{n!}\frac{\Gamma(n+r+1)\Gamma(S+1)\Gamma(n+2S+2)}{\Gamma(S-r+1)\Gamma(r+1)\Gamma(n+r+S+2)}$$
$$\cdot{}_{\lambda+q+3}F_{\nu+p+3}\left(\begin{matrix}\alpha_\lambda, 1+\frac{n}{2}+S, \frac{3}{2}+\frac{n}{2}+S, b_q+S, 1+S\\ \rho_\nu, 1-r+S, a_p+S, n+r+S+2\end{matrix}; x+y\right)\sin(n+2r+1)\theta$$
$$=\frac{\sqrt{\pi}}{2}\sum_{K=0}^{n}\frac{(\sin\theta)^{2S+1}(\cos\theta-1)^K}{K!\,(n-K)!2^K}\cdot\frac{\Gamma(2+2S+n+K)}{\Gamma\left(\frac{3}{2}+K+S\right)}$$
$$\cdot{}_{\lambda+q+2}F_{\nu+p+2}\left(\begin{matrix}\alpha_\lambda, 1+\frac{n+K}{2}+S, \frac{3}{2}+\frac{n+K}{2}+S, b_q+S\\ \rho_\nu, a_p+S, 1+S, \frac{3}{2}+K+S\end{matrix}; (x+y)\sin^2\theta\right),$$

where $0\leq\theta\leq\pi$.

(4.3)
$$\sum_{r=0}^{\infty}\frac{(-1)^r}{n!}[\Gamma(S-r+1)\Gamma(S+r+2)]^{-1}{}_{\lambda+q+5}F_{\nu+p+6}$$
$$\cdot\left(\begin{matrix}\alpha_\lambda, 1+S, \frac{3}{2}+S, 1+S, \frac{\delta_1+\delta_1'-1}{2}, \frac{\delta_1+\delta_1'}{2}\\ \rho_\nu, 1-r+S, a_p+S, 2+r+S, \delta_1, \delta_1', \delta_1+\delta_1'-1\end{matrix}; 4x\right)\sin(2r+1)\theta$$
$$=\frac{\sqrt{\pi}}{2}\left[\Gamma(S+1)\Gamma\left(S+\frac{3}{2}\right)\right]^{-1}(\sin\theta)^{2S+1}$$
$$\cdot{}_{\lambda+q+4}F_{\nu+p+5}\left(\begin{matrix}\alpha_\lambda, 1+S, \frac{3}{2}+S, b_q+S, \frac{\delta_1+\delta_1'-1}{2}, \frac{\delta_1+\delta_1'}{2}\\ \rho_\nu, a_p+S, 1+S, \frac{3}{2}+S, \delta_1, \delta_1', \delta_1+\delta_1'-1\end{matrix}; 4x\sin^2\theta\right),$$

where $0\leq\theta\leq\pi$.

(4.4)
$$\sum_{r=0}^{\infty}(-1)^r[\Gamma(S-r+1)\Gamma(S+r+2)]^{-1}{}_{\lambda+q+3}F_{\nu+p+3}$$
$$\cdot\left(\begin{matrix}\alpha_\lambda, 1+S, \frac{3}{2}+S, b_q+S, 1+S;\\ \rho_\nu, 1-r+S, a_p+S, 2+r+S;\end{matrix}\ x+y\right)\cdot\sin(2r+1)\theta$$
$$=\frac{\sqrt{\pi}}{2}(\sin\theta)^{2S+1}\left[\Gamma(S+1)\Gamma\left(S+\frac{3}{2}\right)\right]^{-1}$$
$$\cdot{}_{\lambda+q+2}F_{\nu+p+2}\left(\begin{matrix}\alpha_\lambda, 1+S, \frac{3}{2}+S, b_q+S\\ \rho_\nu, a_p+S, 1+S, \frac{3}{2}+S\end{matrix}; (x+y)\sin^2\theta\right),$$

where $0\leq\theta\leq\pi$.

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