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INNER AUTOMORPHISMS AND SEMISTABLE AUTOMORPHISMS OF SOME SPLIT GROUP EXTENSIONS

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0. Introduction

An automorphism of a group G which induces an automorphism on a normal subgroup A of G on the one hand and induces the identity automorphism on the factor group G/A on the other is called an *A-semistable automorphism* of G. The set of all A-semistable automorphisms of G forms the *A-semistability group* of G. We are interested in comparing the A-semistability group $SS(G/A)$ of G with the inner automorphism group $J(G)$ of G, where G is a split abelian extension of a cyclic group A. We record some general remarks which are modifications of a well-known result [1 ; p. 106] in section 1, and examine as an application of general remarks every split cyclic extension of every cyclic group to show that for almost all extensions G of A of this kind $SS(G/A)\supsetneqq J(G)$ in section 2.

1. General Remarks

Let A be a group, written additively but not necessarily abelian, and let Π be a group written multiplicatively. The *split extension* $G=(A,\Pi;\phi)$ with operators $\phi\in\mathrm{Hom}(\Pi,\ \mathbf{Aut}(A))$ consists of all pairs (a,x), where $a\in A$ and $x\in\Pi$, with the operation $(a,\ x)+(b,\ y)=(a+\phi(x)b,\ xy)$ written additively but not necessarily abelian.

Let A be abelian, and let Π be arbitrary. A *ϕ-crossed homomorphism* f of Π into A is a mapping of Π into A satisfying $f(xy)=f(x)+\phi(x)f(y)$. The set of all ϕ-crossed homomorphisms of Π into A forms an abelian group $Z^1(\Pi,A;\phi)$ under the composition $(f+g)(x)=f(x)+g(x)$. A *ϕ-principal crossed homomorphism* f_a of Π into A defined by $a\in A$ is a mapping of Π into A satisfying $f_a(x)=\phi(x)a-a$. The set of all ϕ-principal crossed homomorphisms of Π into A forms a subgroup $B^1(\Pi,A;\phi)$ of $Z^1(\Pi,A;\phi)$. We define a homomorphism Φ of the centralizer $C(\phi(\Pi),\mathbf{Aut}(A))$ of $\phi(\Pi)$ in $\mathbf{Aut}(A)$ into the automorphism group $\mathbf{Aut}(Z^1(\Pi,A;\phi))$ of $Z^1(\Pi,A;\phi)$ by $[\Phi(\alpha)](f)=\alpha f$, i.e., the composite "$f$ followed by α," for

$\alpha \in C(\phi(\Pi), \mathbf{Aut}(A))$ and $f \in Z^1(\Pi, A; \phi)$, and then construct the split extension $(Z^1(\Pi, A; \phi), C(\phi(\Pi), \mathbf{Aut}(A)); \Phi)$ with operators $\Phi \in \mathrm{Hom}(C(\phi(\Pi), \mathrm{Aut}(A)), \mathrm{Aut}(Z^1(\Pi, A; \phi)))$. This is the group consisting of all pairs $[f, \alpha]$, where $f \in Z^1(\Pi, A; \phi)$ and $\alpha \in C(\phi(\Pi), \mathbf{Aut}(A))$, with the standard operation of a split extension. The A-semistability group $SS(G/A)$ of $G = (A, \Pi; \phi)$ is isomorphic to this group under the mapping $\omega \mapsto [f, \alpha]$ for $\omega \in SS(G/A)$ where α and f are determined by $\omega(a, 1) = (\alpha(a), 1)$ and $\omega(0, x) = (f(x), x)$. We shall identify ω with $[f, \alpha]$ in this way.

On the other hand, let A be arbitrary and let Π be abelian. Then the inner automorphism group $J(G)$ of G is a subgroup of the A-semistability group $SS(G/A)$ of G.

Now, let A and Π be both abelian. Then an A-semistable automorphism $[f, \alpha]$ of G is an inner automorphism of G if and only if $f \in B^1(\Pi, A; \phi)$ and $\alpha \in \phi(\Pi)$. Since α is already in the centralizer of $\phi(\Pi)$ in $\mathbf{Aut}(A)$, $\alpha \in \phi(\Pi)$ implies that α is in the center of $\phi(\Pi)$.

Finally, let A be cyclic and let Π be abelian. We combine the preceding remarks and record the result as

PROPOSITION (1.1). *Let G be a split extension of a cyclic group A by an abelian group Π with operators $\phi \in \mathrm{Hom}(\Pi, \mathbf{Aut}(A))$. Let $SS(G/A)$ and $J(G)$ designate the A-semistability group of G and the inner automorphism group of G, respectively. Then*

$$SS(G/A) \approx (Z^1(\Pi, A; \phi), \mathbf{Aut}(A); \Phi)$$

with operators

$$\Phi \in \mathrm{Hom}\ (\mathbf{Aut}(A), \mathbf{Aut}(Z^1(\Pi, A; \phi)))$$

defined by $[\Phi(\alpha)]\ (f) = \alpha f$ for $\alpha \in \mathbf{Aut}(A)$ and $f \in Z^1(\Pi, A; \phi)$, where αf is the composite of $f: \Pi \to A$ and $\alpha: A \to A$. Under this isomorphism, $J(G)$ corresponds to $(B^1(\Pi, A; \phi), \phi(\Pi); \Phi')$ where Φ' is the appropriate restriction of Φ.

This is a modified form of Prop. 2.1. in [1 ; p. 106].

2. Application

The order of a group H is written $|H|$. We wish to apply (1.1) to prove

PROPOSITION (2.1). *Let G be a split extension of a cyclic group A by a cyclic group Π with operators $\phi \in \mathrm{Hom}(\Pi, \mathbf{Aut}(A))$. Then the inner automorphism group of G and the A-semistability group of G coincide if and only if one of the following* *three cases takes place:*

$|A|=1$,

$|A|=2$ *and* $|\Pi|$ *is odd,*

$|A|=p^e$, *where* p *is an odd prime and* $e\geq 1$, *and* ϕ *is onto.*

In all other cases, the inner automorphism group of G is properly included in the A-semistability group of G.

We shall use $Z(\infty)$ and $Z(n)$ to designate the additive group of integers and the additive group of integers modulo n, respectively.

LEMMA 1. *Let* ϕ *be a homomorphism of a cyclic group* Π *onto* $\mathbf{Aut}(Z(n))$.

If $n=2$ *and* $|\Pi|$ *is even or infinite,*

$n=4$, *or*

$n=2p^e$ *where* p *is an odd prime and* $e\geq 1$,

then $B^1(\Pi, Z(n);\phi)$ *is a proper subgroup of* $Z^1(\Pi, Z(n);\phi)$. *On the other hand, if*

$n=p^e$ *where* p *is an odd prime and* $e\geq 1$,

then $B^1(\Pi, Z(n);\phi)=Z^1(\Pi, Z(n);\phi)$.

PROOF. Let n be one of the integers in the statement. If t is a generator of Π, there exists a primitive root g modulo n such that $[\phi(t)](a)=ga$ for all $a\in Z(n)$. For every $h\in Z(n)$, there exists a unique ${}_hf\in Z^1(\Pi, Z(n);\phi)$ such that $[{}_hf](t)=h$. Furthermore, ${}_hf\in B^1(\Pi, Z(n);\phi)$ if and only if $(g-1)x\equiv h \bmod n$ has a solution $x\in Z(n)$.

LEMMA 2. *Let* ϕ *be a homomorphism of a cyclic group* Π *onto* $\mathbf{Aut}(Z(\infty))$. *Then* $B^1(\Pi, Z(\infty);\phi)$ *is a subgroup of index* 2 *of* $Z^1(\Pi, Z(\infty);\phi)$.

PROOF. If t is a generator of Π, then for every $h\in Z(\infty)$ there exists a unique ${}_hf\in Z^1(\Pi, Z(\infty);\phi)$ such that $[{}_hf](t)=h$. Furthermore, ${}_hf\in B^1(\Pi, Z(\infty);\phi)$ if and only if h is even.

We are now in a position to finish the

Proof of (2.1). Suppose that every A-semistable automorphism of G is inner. Then by (1.1) we have $Z^1(\Pi, A;\phi)=B^1(\Pi, A;\phi)$ and $\mathbf{Aut}(A)=\phi(\Pi)$, i.e., ϕ is onto. Hence $\mathbf{Aut}(A)$ is cyclic and therefore $A\approx Z(n)$ for some $n=1, 2, 4, 2p^e, p^e$ or ∞, where p is an odd prime and $e\geq 1$. By Lemma 1 and Lemma 2, one of the three cases stated in (2.1) takes place. Conversely, if $n=1$ or if $n=2$ and $|\Pi|$ is odd, then obviously $Z^1(\Pi, Z(n);\phi)=B^1(\Pi, Z(n);\phi)$. If $n=p^e$, where p is an odd prime and $e\geq 1$, and ϕ is onto, then by Lemma 1, $Z^1(\Pi, Z(n);\phi)=B^1(\Pi, Z(n);\phi)$, whence every A-semistable automorphism of G is inner by (1.1).

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REFERENCE

[1] S. MacLane, *Homology*, Springer, Berlin, 1963.