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SOME AVERAGING PROCESS

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In order to give a theoretical definition for a turbulent fluid motion, it is generally said that the velocity of a particle or the pressure at a given point in such a fluid presents "irregular" fluctuations around an average value, both for the time variable and for the space variable. It appears that averages, and averages which are not constant functions, are here essential. Obviously, random functions are well suited to the search for such averages and a great number of investigations concerning turbulent fluid motions use probability theory, and therefore mathematical expectations as averaging operators. This means that averages are computed via many different experiments done at random. Another point of view, historically the first, was to study averages along a time variable by using an expression such as $\frac{1}{2T}\int_{-T}^{+T} f(t)\, dt$ (and its limit when T increases) or to study averages along a space variable by using similar integrals. Naturally, the link between these two investigations is to be found in ergodic theorems. However, following this latter averaging approach we may look for the axiomatic rules to be satisfied by what shall be considered as an average for a function. Because much freedom remains, we may require the linearity of the correspondence between f and its average Pf. We also may ask for some assumption of continuity for the linear operator P (rather than any assumption of positivity). There is a need for a supplementary property since linearity and continuity are far too general to provide a means of obtaining useful averaging methods. An idea, originated by O. Reynolds (cf. [1]), is to look for an operator P "commuting" with the differential operator governing fluid motion, namely the Navier-Stokes equation. Recall that the vectorial Navier-Stokes equation, valid for a newtonian fluid, can be written as

$$(1) \qquad \rho\frac{\partial\vec{V}}{\partial t}=\rho\vec{f}-\text{grad } p+\nu\rho\Delta(\vec{V})-\rho \text{ div}(\vec{V}\otimes\vec{V})$$

where $\vec{V}(t,M)$ is the velocity at a point M and at time t, the components of which are V_1, V_2 and V_3.

In Eq. (1), $\Delta\vec{V}(t,M)$ is a vector whose components are $\Delta V_1, \Delta V_2$ and ΔV_3, p is the pressure, $\vec{f}$ an external force

and div $(\vec{V}\otimes\vec{V})$ is a non linear term, which denotes a vector whose components are equal to $\sum_{j=1}^{3}\frac{\partial}{\partial x_i}(V_iV_j)$ for $i=1,2$ and 3.

Eventually, for incompressible fluids, we must add

(2) $$\operatorname{div}\vec{V}=0$$

O. Reynolds looked for an operator P acting on $\vec{V}$ and p in such a way that $P(\vec{V})$ satisfies a Navier-Stokes equation with a supplementary term, the turbulent one, considered as added to the external force

(3) $$\rho\frac{\partial P(\vec{V})}{\partial t}=P[\vec{f}-\operatorname{div}((\vec{V}-P(\vec{V}))\otimes(\vec{V}-P(\vec{V})))]+\nu\rho\Delta P(\vec{V})$$
$$-\rho\operatorname{div}(P(\vec{V})\otimes P(\vec{V}))-\operatorname{grad}(P(p))$$

It has been proved (cf. [1] M. L. Dubreil-Jacotin) that if P acts on variable t only, then (3) is a consequence of (2) if we add to some continuity and stationary assumptions on P, which means commutation with translations, the equation:

$$P(f^2)=(Pf)^2-P(f-Pf)^2$$

This leads to the functional equation characterizing Reynolds operators which is

(4) $$P(f\,Pg+g\,Pf)=Pf\,Pg+P(Pf\,Pg)$$

Reynolds operators have been studied by many authors (cf. [1] G.C. Rota).

In this work, we present a generalization of these operators, the so-called $D(\alpha)$ operators. In order to obtain interesting representation theorems, we restrict ourselves to special functions, e.g. periodic ones, or more generally almost-periodic functions.

This paper is divided into

- I Reynolds Operators over periodic functions
- II Operators of type $D(\alpha)$ over periodic functions
- III Operators of type $D(1)$ over periodic functions
- IV Generalization to almost-periodic functions
- V A need for other generalizations

On rather particular functional spaces, we shall try to distinguish between the averaging properties and the smoothing properties which are both consequences of the functional equation satisfied by Reynolds operators or more generally by $D(\alpha)$ operators.

In the sequel, the space of all continuous functions taking complex values and defined over a topological compact space X, will be denoted by $C(X)$, its elements being denoted by f, g etc. A linear operator on $C(X)$ will be denoted by P or R. Generally, $\wedge$ will be a sub-semigroup of the additive group Z of all relative integers. Z^+ is the sub-semigroup of all positive or zero integers, Z^- is the sub-semigroup of all negative or zero integers. We denote by I the identity operator on $C(X)$. Some of the results given in this paper have previously been announced (cf. [4] J. G. Dhombres).

I. Reynolds Operators Over Periodic Functions

Let k be any integer. We denote by $C(T_k)$ the algebra of all continuous $2\pi/k$-periodic functions defined over R and taking complex values. For $k=0$, $C(T_0)$ is simply the set of all constant functions. On $C(T_k)$ we set up the uniform norm.

For every real number h, operator T_h represents the translation operator:

$$T_h f : x \to f(x+h)$$

A linear operator $P: C(T_k) \to C(T_k)$ is *stationary* when, for every real number h, we have the commuative property

$$P(T_h(f)) = T_h(P(f)) \tag{5}$$

In order to describe all continuous and stationary Reynolds operators, we must begin with some definitions.

Let k be any integer

(*a*) *For* $k=0$, *we define* $P_0 f = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(t)\, dt$

(*b*) *For* $k \neq 0$, *we define* $P_k f(x) = \dfrac{f(x) + f(x+\frac{2\pi}{k}) + \cdots + f(x+\frac{k-1}{k}2\pi)}{k}$

Operator P_k is defined over $C(T_1)$ and takes its values in $C(T_k)$.

Now let s be any complex number (finite or not) such that s is different from a multiple of k. We define R_s, from $C(T_k)$ into $C(T_1)$ by

(*c*) $R_s f(x) = \dfrac{s}{2\sin\frac{\pi s}{k}} \displaystyle\int_{-\frac{\pi}{k}}^{+\frac{\pi}{k}} e^{-its} f(x-t-\frac{\pi}{k})\,dt$

and for the case where s is equal to ∞:

(*d*) $R_\infty f(x) = f(x)$ $\qquad R_\infty$ is the *identity operator.*

THEOREM 1. *A non-zero stationary and continuous operator over* $C(T_1)$ *is a Reynolds operator if and only if there exist an integer k and a complex number s,* *different from a multiple of k, such that*

$$P=R_s \circ P_k$$

PROOF. Consider functions e_n defined by $e_n: x \to e^{inx}$ where n is any relative integer. These functions are the only eigen-functions for all operators T_h and, due to the commutation property (5) for P and T_h, are also eigen-functions of P, which gives:

(6) $$P(e_n)=a(n)e_n.$$

Compute Reynolds relation with $f=e_n$ and $g=e_m$

(7) $$P(e_n P(e_m)+e_m P(e_n))=P(e_n)P(e_m)+P(P(e_n)P(e_m))$$

which implies a relation for the function $n \to a(n)$ defined over Z:

(8) $$a(n+m)(a(n)+a(m))=a(n)a(m)+a(n)a(m)a(n+m)$$

Now define a subset $\wedge$ of Z by $\wedge=\{n \mid n \in Z: a(n) \neq 0\}$.

0 belongs to $\wedge$, because $2a^2(0)=a^2(0)+a^3(0)$ yields $a(0)=0$ or $a(0)=1$. But $a(0)=0$ implies $a(n)=0$ for all n and so $P \equiv 0$ which is excluded. Therefore $a(0)=1$.

Suppose $a(n+m)=0$, then $a(n)a(m)=0$ and this proves that if n and m are in $\wedge$, so is $n+m$.

Suppose n is in $\wedge$, then $(a(n)+a(-n))=2a(n)a(-n)$ which proves that $a(-n)$ is different from 0 and so $-n$ is in $\wedge$.

Finally, we have proved that $\wedge$ is a subgroup of Z. Therefore, there exists an integer k such that $\wedge=kZ$.

Relation (8), for n and m restricted to $\wedge$, provides

$$\frac{1}{a(n)}+\frac{1}{a(m)}=\frac{1}{a(n+m)}+1$$

Defining $b(n)=\frac{1}{a(n)}-1$ which is possible for n in $\wedge$, we get the familiar functional equation

(9) $$b(n+m)=b(n)+b(m)$$

As n and m are multiples of k, we then get

$$b(n)=\frac{n}{k}b(k)$$

where $b(k)$ is a complex number such that $b(n)+1$ is different from zero. This implies $b(k) \neq -\frac{k}{n}$ for all n which are non-zero multiples of k.

(a) The case $b(k)=0$ implies $a(n) \equiv 1$ for n in $\wedge$. According to the continuity of the operation P, if f is expanded in Fourier series along

$$f \sim \sum_n c_n e^{inx} \qquad \text{where} \qquad c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(t)e^{-int}\,dt$$

then Pf, in turn, possesses the following Fourier expansion

$$Pf \sim \sum_n c_{nk} e^{inkx}$$

If $k=0$, we get directly $Pf = c_0(f)$, that is $Pf = P_0 f = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(t)\,dt$ according to the notation defined just before theorem 1.

If k is different from 0, we see that $P = P_k$ where

$$(10) \qquad P_k(f)(x) = \frac{f(x) + f\left(x + \frac{2\pi}{k}\right) + \cdots + f\left(x + \frac{k-1}{k}2\pi\right)}{k}$$

and we note that $P_1(f) = f$, that is P_1 is the identity operator.

(b) The second case is $b(k) \neq 0$ and we suppose first that $k=1$. We can then define $b(1) = \frac{1}{s}$ where s is a complex number but not a relative integer. Then the Fourier expansion of Pf is given by

$$(11) \qquad Pf \sim \sum_n c_n \frac{1}{\left(n \cdot \frac{1}{s} + 1\right)} e^{inx}$$

because $a(n) = \left(n \cdot \frac{1}{s} + 1\right)^{-1}$ for n in $\wedge = Z$ due to equation (9).

Then, taking derivatives in the sense of distribution, we get the following differential equation concerning Pf:

$$(12) \qquad \frac{1}{is}\,\frac{d(Pf)}{dx}(x) + Pf(x) = f(x)$$

Using the fact that Pf must belong to the algebra $C(T_1)$ we find, after some computation, a solution for (12):

$$Pf(x) = R_s f(x) = \frac{is}{1 - e^{-2i\pi s}}\int_0^{2\pi} e^{-ist} f(x-t)\,dt = \frac{s}{2\sin\pi s}\int_{-\pi}^{+\pi} e^{-its} f(x-t-\pi)\,dt$$

(c) When $k \neq 1$, then we first use P_k from $C(T_1)$ into $C(T_k)$ and then compose it with R_s from $C(T_k)$ into $C(T_k) \subset C(T_1)$ in order to obtain the operator P, the image of which is included in $C(T_k)$

$$P = R_s \circ P_k$$

This comes from the fact that we have obtained the following Fourier series for Pf:

(13) $$Pf \sim \sum_n c_{nk} \frac{1}{\frac{nk}{s}+1} e^{inkx}$$

because we have taken $b(k)=\frac{k}{s}$ and so s must not be a multiple of k. Conversely, it is possible to verify that any operator like $P_k = R_\infty \circ P_k$ or $R_s \circ P_k$ is a continuous Reynolds operator which commutes with translation operators on the algebra $C(T_1)$.

COROLLARY 1. *An idempotent, non-zero, stationary and continuous Reynolds operator on* $C(T_1)$ *is of the form* P_k *as given by theorem* 1.

It suffices to consider the action of P on the Fourier expansion of function f.

Operators like P_k appear as typical averaging operators, the average being taken along an arithmetical progression. It is possible to prove directly that P_k satisfies the following equation

(14) $$P_k(fP_k(g)) = P_k(f)P_k(g)$$

which is the functional equation characterizing *semi-multiplicative symmetric operators*. Equation (14) tells us that $P_k(g)$ behaves as a constant for operator P_k, which appears intuitively to be an averaging property.

Conversely, all non-zero stationary and continuous operators on $C(T_1)$ satisfying equation (14) are of the form P_k. More generally, following the same lines, we can prove

PROPOSITION 1. *A bounded linear operator* P *on* $C(T_1)$ *such that* $P(1)=1$ *and which is multiplicatively symmetric, that is which satisfies* $P(fPg)=P(gPf)$ *for all* f *and* g *in* $C(T_1)$, *is stationary if and only if it is of the form* P_k *of theorem* 1.

The intuitive notion of average can now be made mathematically precise, by using equation (14) or only the multiplicatively symmetric relation in Proposition 1 (cf. [1], [2] J. G. Dhombres).

Obviously, with averaging properties, P_k also manifests smoothing properties. For example, if f is of bounded variation, then $V(P_k f) \leq V(f)$, where V denotes the total variation of f.

Theorem 1 asserts that P is a Reynolds operator if it is made up of the composition of an operator P_k and an operator R_s. This last operator multiplies the amplitude c_n of the n-th harmonic e_n by $(n/s+1)^{-1}$. This means that R_s also has a smoothing property in the sense that $R_s f$ is more regular than f because its n-th Fourier coefficient is more quickly converging towards zero when n tends to infinity. For example, $R_s(f)$, for f in $C(T_1)$, always possesses an absolutely convergent Fourier series as can be seen by the following inequality

$$\sum_n \left| \frac{c_n(f)}{\frac{n}{s}+1} \right| \leq (\sum_n |c_n(f)|^2)^{\frac{1}{2}} \left(\sum_n \frac{1}{\left|\frac{n}{s}+1\right|^2} \right)^{\frac{1}{2}} \tag{15}$$

We have seen that operators P_k possess both averaging and smoothness properties. However, R_s has only a smoothing property, according to the following easy corollary:

COROLLARY 2. *A continuous and stationary Reynolds operator P over $C(T_1)$ is one-to-one if and only if P is an operator of type R_s.*

Let us now give a quantitative measurement of the smoothness of R_s as an operator from $C(T_1)$ into $C(T_1)$.

If f has Fourier coefficients such that $\sum_n |c_n(f)|$ is a convergent series, then we have

$$V(R_s f) \leq \frac{2\pi}{\varepsilon} \sum_n |c_n(f)|$$

where $V(R_s f)$ denotes the total variation of $R_s f$ and ε is defined by

$$\operatorname*{Inf}_{n\in Z} \left| \frac{1}{n} + \frac{1}{s} \right| = \frac{1}{\varepsilon}.$$

If f itself is of bounded variation, then

$$V(R_s f) \leq A(s) V(f)$$

and $A(s)$ is a constant depending upon s only. Obviously $A(s) = \|R_s\|$ which is the norm of the bounded operator R_s for the uniform norm (Note, by contrast that $\|P_k\| = 1$).

Operator R_s transforms real functions into real functions if and only if s is a purely imaginary number, let us say $s = is'$. In this last case $\|R_s\| = 1$ and even R_s is a positive operator, As a by-result, we note that if P is a linear Reynolds operator satisfying the hypothesis of theorem 1, and transforms real functions into real functions, then P is a positive operator because R_s and P_k are positive operators under these conditions.

Writing $s = s_1 + is_2$, we find from a simple computation

$$\text{when } s_2 \neq 0, \text{ then } \|R_s\| = \left\{ \frac{\left(\frac{s_1}{s_2}\right)^2 + 1}{\frac{\sin^2 \pi s_1}{sh^2 \pi s_2} + 1} \right\}^{\frac{1}{2}} \geq 1 \tag{16}$$

and so we can find an operator R_s, the norm of which is as near to 1 as we wish,

(17) when $s_2=0$, then $\|R_s\|=\left|\dfrac{\pi s_1}{\sin\pi s_1}\right|>1$ for $s_1\neq 0$.

Naturally, if P and P' are operators satisfying the hypothesis of theorem 1, then $P\circ P'=P'\circ P$. However the commutative product $R_s\circ R_t=R_t\circ R_s$ is not a Reynolds operator by opposition with $P_k\circ P_l=P_m$, where m is the lowest common multiple of k and l.

Obviously, theorem 1 remains true if we replace the continuity of the operator P for the uniform norm by the continuity of P for a norm L_p, with $p\geq 1$, or any functional norm for which $f\to c_n(f)$ are continuous linear forms.

II. Operators of Type $D(\alpha)$ Over Periodic Functions

We have introduced Reynolds operators via the Navier-Stokes equation and the notion of a mean. An algebraical study of this notion leads to other types of operators which have been gathered under the name of "multiplicatively related" operators (cf. [2] J.G. Dhombres). A subclass of such operators Q generalizes Reynolds operators: they are called $D(\alpha)$ or $D'(\alpha)$ operators and will be studied now:

$D(\alpha)$ $\qquad Q(fQg+gQf)=\alpha Q(fg)+(1-\alpha)QfQg+Q(Qf\cdot Qg)$

$D'(\alpha)$ $\qquad Q(fQg+gQf)=Q(fg)+(1-\alpha)QfQg+\alpha Q(QfQg)$

A multiplication by α^{-1} exchanges types $D(\alpha)$ and $D'(\alpha)$ whereas $D(0)$ is a *Reynolds operator* and $D'(0)$ is a *Baxter operator*. We shall call operator D an operator of type $D(1)$.

THEOREM 2. *Let α be a real number such that $\alpha\neq 1$ and $0<\alpha<2$. Let Q be a bounded linear stationary operator over $C(T_1)$ of type $D(\alpha)$. Suppose moreover that $Q(1)=1$ and that Q transforms real functions into real functions. Then there exists a real number β and*

$$Qf(x)=\alpha\sum_{k=0}^{\infty}(1-\alpha)^k f(x+k\beta)$$

Conversely, this last equation furnishes a linear operator of type $D(\alpha)$.

PROOF. With notations of §2, we get the functional equation satisfied by $n\to a(n)$, where $a(n)$ is the eigenvalue of e_n for operator P_n:

(18) $\qquad (a(n)+a(m))a(n+m)=\alpha a(n+m)+(1-\alpha)a(n)a(m)+a(n+m)a(n)a(m)$

We now define three disjoint subsets of $\mathbf{Z}$:

$$\wedge=\{n\,|\,n\in Z;\, a(n)\neq 0 \text{ and } a(n)\neq\alpha\}$$

$\wedge_0=\{n|n\in Z;a(n)=0\}$

$\wedge_\alpha=\{n|n\in Z;a(n)=\alpha\}$

$0\in\wedge$ due to our hypothesis $Q(1)=1$. If $a(n+m)=0$, then $a(n)a(m)=0$ and if $a(n+m)=\alpha$, then $(\alpha-a(n))(\alpha-a(m))=0$. These two properties imply that if n and m are in $\wedge$, $(n+m)$ is also in $\wedge$. In addition, we get $a(-n)=\frac{\alpha-a(n)}{1-(2-\alpha)a(n)}$, so that $-n\in\wedge$ as soon as n does, using the fact that $\alpha\neq0$ and $\alpha\neq1$.

Finally, $\wedge$ is a subgroup of Z and so $\wedge=kZ$ for some integer k. We also notice that $\wedge_\alpha=-\wedge_0$. Moreover, if n and m belong to $\wedge_0$, then $\alpha a(n+m)=0$ and so $\wedge_0$ is a semi-group in Z. But if we suppose that 1 does not belong to $\wedge$, then it belongs to either $\wedge_0$ or $\wedge_\alpha$ and as k belongs to $\wedge$, we must have $k=0$. There are two cases:

$k=0$. $\wedge_0=Z^-\setminus[0]$ and $\wedge_\alpha=Z^+\setminus[0]$ if we suppose, for example, that 1 belongs to $\wedge_\alpha$. With a function f in $C(T_1)$, we associate its Fourier expansion

$$f\sim\sum_n c_n e^{inx}$$

Due to the assumed continuity of Q, we get the following Fourier expansion for Qf:

$$Qf\sim c_0+\alpha\sum_{n>0}c_n e^{inx}$$

But an operator R defined by $f\to Rf\sim\sum_{n\geq0}c_n e^{inx}$ is not bounded for the uniform norm according to a theorem due to M. Riesz (cf. [1] A. Zygmund). This theorem tells us that case $k=0$ cannot happen.

$k\neq0$. Then $k=1$ and $\wedge=Z$. We rewrite equation (18) and after having defined $b(n)=\frac{\alpha}{\alpha-1}\left(\frac{1}{a(n)}-\frac{1}{\alpha}\right)$, we find a simpler equation

$$b(n+m)=b(n)b(m) \tag{19}$$

But to equation (19), we must add $b(n)\neq0$ and $b(n)\neq(1-\alpha)^{-1}$ for all n. We get, solving (19), $b(n)=a^n$ where $a=b(1)$, and so

$$a(n)=\left(\left(1-\frac{1}{\alpha}\right)a^n+\frac{1}{\alpha}\right)^{-1}$$

Due to the continuity of operator Q, we get for the Fourier expansion of Qf

$$Qf\sim\sum_n\left(\left(1-\frac{1}{\alpha}\right)a^n+\frac{1}{\alpha}\right)^{-1}c_n e^{inx} \tag{20}$$

As stated in the hypothesis of theorem 2, we assume that f conserves real functions, so that, for all n

$$\overline{a(n)}=a(-n) \text{ and so } |a|=1$$

We then define $a=e^{i\beta}$ where β is a real number and from (20) derive easily a difference equation concerning Qf which we now denote by $Q_{\beta,\alpha}f$

$$(21) \qquad \left(1-\frac{1}{\alpha}\right)Q_{\beta,\alpha}f(x+\beta)+\frac{1}{\alpha}Q_{\beta,\alpha}(x)=f(x)$$

This difference equation has at most one solution in $C(T_1)$, except when for an integer n, $\left(1-\frac{1}{\alpha}\right)e^{in\beta}+\frac{1}{\alpha}=0$. But as α is real, and different from zero, the exceptional cases are $\alpha=2$ with $\beta=\frac{\pi}{n}(\text{mod } 2\pi/n)$. If then $0<\alpha<2$ and $\alpha\neq 1$, equation (21) has a solution given by the following absolutely convergent series

$$Q_{\beta,\alpha}f(x)=\alpha\sum_{k=0}^{\infty}(1-\alpha)^k f(x+k\beta)$$

which we obtained after having taken the inverse of $\frac{1}{\alpha}(\delta_0-(1-\alpha)\delta_\beta)$ in the convolution algebra of bounded Radon measures on T_1. This ends the proof of theorem 2.

NOTE 1. Suppose $\beta=\alpha\eta$, then equation (21) can also be written as

$$\frac{Q_{\beta,\alpha}f(x)-Q_{\beta,\alpha}f(x+\alpha\eta)}{\alpha}+Q_{\beta,\alpha}f(x+\alpha\eta)=f(x)$$

When α tends to zero, we formally get a differential equation for $Pf=\lim_{\alpha\to 0} Q_{\beta,\alpha}f$

$$(22) \qquad -\eta\frac{dPf}{dx}(x)+Pf(x)=f(x)$$

and so P appears as a Reynolds operator $P_{i\eta}$ according to notation of § 1 (cf. Equation (12)). This result might have been foreseen because $D(0)$ operator is a Reynolds operator.

NOTE 2. If β is not an ergodic element of $\mathbf{R}$ (mod 2π), for example $\beta=-\frac{2\pi}{n}$ then we find that $Q_{\beta,\alpha}f$ is a relatively usual weighted mean

$$Q_{\beta,\alpha}f(x)=\frac{\alpha}{1-(1-\alpha)^n}\left[f(x)+(1-\alpha)f\left(x-\frac{2\pi}{n}\right)+\cdots+(1-\alpha)^{n-1}f\left(x-\frac{n-1}{n}2\pi\right)\right]$$

(When α tends to zero, $Q_{\beta,\alpha}f(x)$ formally tends towards $P_n f(x)$ where P_n is the averaging operator occurring in theorem 1).

If β is an ergodic element (mod 2π), that is if $[e^{ik\beta}]_{k\in Z}$ is dense in the unit circle, then $Q_{\beta,\alpha}f(x)$ appears to be the Abel summation process of the divergent series $\sum_{k=0}^{\infty}f(x+k\beta)$. For a given β, $\alpha\to Q_{\beta,\alpha}f$ is an analytic function of α.

COROLLARY 3. *Suppose operator Q satisfies the hypothesis of theorem* 2 *with* $0<\alpha<1$. *Then, Q is of type* $Q_{\alpha,\beta}$ *for an ergodic* β *if and only if* $Qf(0)=0$ *implies* $f\equiv 0$ *for a positive function f.*

If β is ergodic, and $0<\alpha<1$, $Qf(0)=0$ implies $f(k\beta)=0$ for all relative integers k, so that $f\equiv 0$. The converse is easily derived.

We get $\|Q_{\alpha,\beta}\|=1$ for all β and if f is of bounded variation, we get the following smoothing property

$$V(Q_{\alpha,\beta}f)\leq V(f)$$

However, $Q_{\alpha,\beta}$ cannot be used as an averaging operator because of the following result:

COROLLARY 4. *An operator satisfying the hypothesis of theorem* 2 *is a bijective operator.*

Equation (21) also yields for the uniform norm

$$\|Q_{\beta,\alpha}f\|=\|f\| \quad \text{for} \quad 1<\alpha<2 \tag{23}$$

and

$$\left(\frac{2}{\alpha}-1\right)^{-1}\|f\|\leq\|Q_{\beta,\alpha}f\|\leq\|f\| \text{ for } 0<\alpha<1 \tag{24}$$

Due to (21), operator $Q_{\beta,\alpha}$ appears as a multiple of the resolvant of a convolution operator. This operator being the convolution by a Dirac measure δ_β

$$Q_{\beta,\alpha}=\alpha(\delta_0+(\alpha-1)\delta_\beta)^{-1}$$

Such a Dirac measure δ_β induces a convolution operator M according to $Mf=\delta_\beta * f$, and M is a multiplicative operator:

$$M(fg)=(Mf)(Mg) \tag{25}$$

This last result immediately leads to this generalization. Let M be a bounded linear operator on $C(T_1)$, satisfying (25). We suppose that $(\alpha-1)$ does not belong to the spectrum of M. Consider an operator Q_α defined by $Q_\alpha=a(I+(\alpha-1)M)^{-1}$ which gives:

$$M=\frac{I-\alpha Q_\alpha^{-1}}{1-\alpha} \tag{26}$$

Starting from $M(Q_\alpha f\cdot Q_\alpha g)=M(Q_\alpha f)M(Q_\alpha g)$, we get

$$Q_\alpha f Q_\alpha g-\alpha(fQ_\alpha g+gQ_\alpha f)+\alpha^2 fg=(1-\alpha)(Q_\alpha fQ_\alpha g-\alpha Q_\alpha^{-1}(Q_\alpha fQ_\alpha g))$$

and finally Q_α appears as an operator of type $D(\alpha)$, when we exclude $\alpha=0$ and $\alpha=1$. The converse statement is also true, namely if Q_α is an operator of type D, for which 1 does not belong to the spectrum, then (26) furnishes a bounded and multiplicative linear operator.

THEOREM 3. *Let Q be a bounded linear operator on $C(T_1)$. Suppose* $Q(1)=1$ *and suppose that Q possesses a bounded inverse. Operator Q is of type $D(\alpha)$ for* $0<\alpha<2$ *and* $\alpha \neq 1$ *if and only if there exists a continuous 2π-periodic function $\Phi: R \to R$ such that*

$$Qf=\alpha\sum_{k=0}^{\infty}(1-\alpha)^k f\circ\Phi^{(k)}$$

where $\Phi^{(k)}$ denotes the composition of k-times Φ by itself.

First, suppose that Q is $D(\alpha)$. Then $M=\dfrac{I-\alpha Q}{1-\alpha}$ is a bounded multiplicative operator on $C(T_1)$ such that $M(1)=1$. Then, for every x_0 in T_1, $f\to Mf(x_0)$ is a continuous and multiplicative form on $C(T_1)$ and so $Mf(x_0)=f(y_0)$ according to a well-known result. (cf. [1] N. Dunford-J.T. Schwartz). We write $y_0=\Phi(x_0)$ and using the continuity of operator Q_α, we deduce the continuity of Φ. Then $Mf=f\circ\Phi$. But we also have

(27) $$(1-\alpha)M=I-\alpha Q^{-1}$$

which yields $MQ=QM$, that is

(28) $$Q(f\circ\Phi)=Qf\circ\Phi$$

Finally we get the following functional equation, by using (27) and (28)

(29) $$\left(1-\frac{1}{\alpha}\right)Qf\circ\Phi+\frac{1}{\alpha}Qf=f$$

The unique solution in $C(T_1)$, within $0<\alpha<2$, is

(30) $$Qf=\alpha\sum_{k=0}^{\infty}(1-\alpha)^k f\circ\Phi^k$$

Conversely (30) defines an operator $D(\alpha)$ satisfying the hypothesis of theorem 3. We notice that Q is an isometry for $1<\alpha<2$ and a positive operator for $0<\alpha<1$ and that inequalities (24) remain valid.

III. Operators of Type D Over Periodic Functions

This type is in fact the only remaining case to be studied, even if we do not suppose $Pe=e$, as a detailed analysis would reveal:

(31) $$P(fPg+gPf)=P(fg)+P(PfPg)$$

With our previous notations, we get a functional equation for $a(n)$:

(32) $$a(n+m)(a(n)-1)(a(m)-1)=0$$

which yields $Pe=e$ or $Pe=0$.

First Case: $Pe=e$. As in § 2, we define $\wedge$, $\wedge_0$ and $\wedge_1$ as subsets of Z. Equation

(32) proves that both $\wedge_0$ and $\mathscr{C} \wedge_1$ are semi-groups of $\boldsymbol{Z}$. Moreover if $a(2n)$ is different from zero, then $a(n)=1$ and if $a(2n)=0$, we get $a(-2n)=1$ and so $a(-n)=1$. This proves that, for any n, either n or $-n$ belongs to $\wedge_1$. Such features distinguish type D. Subsets $\mathscr{C} \wedge_1$ and $\wedge_0$ are semi-groups of $\boldsymbol{Z}$ which do not contain 0 and so they are both contained in $\boldsymbol{Z}^+$ or in $\boldsymbol{Z}^-$. For example, for a positive integer n_0, we may take $\wedge_1=]-\infty, n_0]$; $\wedge_0=[2n_0, \infty[$ and leave the values of $a(n)$ to be arbitrary in $\wedge=]n_0, 2n_0[$. Such operators are common in harmonic analysis for computing means of Fourier series. For instance, let P_{n_0} be an operator D such that the values of $a(n)$, for n between n_0 and $2n_0$, are located on the straight line between $a(n_0)=1$ and $a(2n_0)=0$. Let P' be an operator, also of type D for which $a'(n)=a(-n)$. Then, operator $V_{n_0}=P_{n_0} \circ P'_{n_0}=P'_{n_0} \circ P_{n_0}$ is the de la Vallee Poussin operator (cf. [1] A. Zygmund). In the same way, we may find Dirichlet operators. It is not the place here to exhibit properties of these usual summation operators which can be deduced directly from the functional equation (31).

Second Case: $Pe=0$. Equation (31) yields $P^2=P$ and we check that $\wedge=\phi$. It appears that $\wedge_0$ is semi-group of $\boldsymbol{Z}$ containing 0 and this fact is distinctive. We must notice the interesting particular case of an idempotent Baxter operator which satisfies a functional equation, simpler than (31).

$$P(fPg+gPf)=P(fg)+PfPg \tag{33}$$

For such an operator, $\wedge_1$ is also a semi-group and we find (cf. [2] J.G. Dhombres)

$$\wedge_1=\boldsymbol{Z}^+ \text{ or } \wedge_1=\boldsymbol{Z}^-$$

In the first case for example, we get

$$Pf\sim\sum_{n\geq 0}c_n e^{inx}$$

However, M. Riesz' theorem ascertains that such an operator is not continuous for the uniform norm, being continuous for a norm $L^p(\boldsymbol{T}_1)$ when $+\infty>p>1$.

IV. Generalization to Almost-Periodic Functions

Starting from a turbulent fluid, it might have appeared strange to restrict ourselves to periodic functions. In fact, what we needed was to be able to use Fourier analysis as a tool. We can extend our methods to the different kinds of almost periodic functions, that is to functions which are the limits for different norms, of generalized trigonometric polynomials like $\sum_{n=n_1}^{n=n_2}c_n e^{i\lambda_n x}$ where λ_n is a real number. More generally, we shall use some particular results of abstract harmonic analysis, which are summarized in [1] J.G. Dhombres, where references are given.

Let G be an abelian, locally compact topological group. We use $\hat{G}$ to denote its dual, that is the group of all continuous characters of G with the open-compact topology. A continuous function taking complex values on G is almost periodic if it is the uniform limit of generalized trigonometrical polynomials like $\sum_{n=n_1}^{n=n_2} c_n\langle x, \hat{x}_n\rangle$. With each such function is associated its unique generalized Fourier series:

$$f \sim \sum_{\hat{x}_n \in \hat{G}} c_n \langle x, \hat{x}_n\rangle$$

Another characterization of such almost periodic functions f is that the set $[T_h f]_{h\in G}$ of translates of f is a relatively compact subset for the topology of uniform norm on $C(G)$. The set of all almost-periodic functions constitutes a C^*-algebra, which we denote by $A(G)$.

Algebraically and topologically, $A(G)$ is isomorphic with the algebra of all continuous functions over $\overline{G}$ where $\overline{G}$ is a compact group containing G as a dense subgroup. $\overline{G}$ is the Bohr compactification of G.

Continuous linear operators of type $D(\alpha)$, which commute with the translations generated by G on $A(G)$, can be investigated along the same lines as in the case where $G=T_1=\overline{G}$. Each character of G is an eigen-function for P, that is $P(\hat{x}) =a(\hat{x})\hat{x}$. We replace the functional equation $D(\alpha)$ by a functional equation concerning $a(\hat{x})$. Obviously, for general $\hat{G}$, subgroups (like $\wedge$) have not the simple aspect of the subgroups of Z and generally we cannot use differential equations like in the Reynolds case on $C(T_1)$. Moreover representation formulae like those giving R_s or P_k, or like formula in Theorem 2, are not easy to discover. Various algebraical or topological assumptions can be made for G, $\hat{G}$ or P in order to diminish the number of different possible cases. We shall list here some results without proofs. In the isomorphism between $A(G)$ and $C(\overline{G})$, let $b(f)$ be the image of f. We get, for $+\infty\geq p\geq 1$, a norm on $A(G)$ by defining:

$$M_p(f)=\Big(\int_{\overline{G}}\Big|b(f)(\overline{x})\Big|^p d\overline{x}\Big)^{1/p} \quad \text{and} \quad M_\infty(f)=\sup_{x\in G}|f(x)|$$

An *analytical operator* P is, by definition, such that its range cannot simultaneously contain f and $\bar{f}$ (complex conjugacy) unless f is a constant.

THEOREM 4. *There exists a bijection between the set of all semi-groups $\wedge$ of $\hat{G}$ which transform $\hat{G}$ into an ordered group and the stationary Baxter operators P on $A(G)$ which are continuous for M_p with $\infty>p>1$. More specifically, let $\wedge$ be* *such a semi-group, then*

$$Pf\sim \sum_{\hat{x}_n\in\wedge} C(\hat{x}_n, f)\langle x, \hat{x}_n\rangle$$

where $c(\hat{x}_n, f)$ *is the generalized Fourier coefficient of* f *at* $\hat{x}_n$.

THEOREM 5. *There exists a bijection between the continuous functions* Φ *from* $\bar{G}$ *into* $\bar{G}$ *and the stationary operators* P *of type* $D(\alpha)$ (*with* $0<\alpha<1$ *and* $P(1)=1$) *which possess a bounded inverse over* $A(G)$ *when* G *is an abelian locally compact group. More specifically,*

$$b(Pf)=\alpha\sum_{k=0}^{\infty}(1-\alpha)^k b(f)\circ\Phi^{(k)}$$

where $\Phi^{(k)}$ *denotes the composition of k-times* Φ *by itself.*

V. A Need for Other Generalizations

From the point of view of functional analysis, operators acting as means, or averaging operators, must be defined on functional spaces. We have tried to show in the preceding sections that interesting representation theorems can be obtained for operators $D(\alpha)$, which are defined by some functional equation, when they operate on algebras of almost-periodic functions. In retrospect, after our introduction using the Navier-Stokes equation and related non-linear equations, our investigations make sense if such equations possess, for some common boundary conditions, solutions which are almost-periodic. This is the case and there exist nowadays numerous results of that kind. This is not the place here to quote the precise theorems and we shall just give a recent reference ([1] L. Amerio, J. Prouse). However, a rather technical example will be detailed in a forthcoming paper ([3] J.G. Dhombres). It must also be added that from a practical point of view, we can expect, and sometimes may prove, that the operators previously exhibited behave as averaging operators, even for functions which are not almost-periodic, and that our representation formulae are approximatively correct.

On another front, special functions, the so-called pseudo-random functions ("fonctions pseudo-aléatoires") have been investigated specially for the study of turbulent fluid motion (cf. [1] J. Bass). Nowadays, various representation theorems and natural generalizations are known. Moreover, it has been proved that some non-linear partial differential equations, including the Navier-Stokes equation, possess solutions which are pseudo-random functions. (For recent results, and various references on those functions, we refer to [2] J. Bass). These pseudo-random functions, which we choose to take on the real axis to make things simpler, are defined by the following asymptotic property:

$$\lim_{h\to+\infty}\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T} f(t+h)\overline{f(t)}dt\right)=0$$

Eventually we may add a regularity assumption like continuity. This definition means that the correlation between f and $T_h f$ is small if h is big enough. Such functions may have an irregular local aspect but may retain a certain kind of periodical behaviour just like the turbulent solution of the Navier-Stokes equation. To compute means on such functions is obviously a natural but unfortunately an arduous task. In fact, very few results are so far available, in part because known representations of usual pseudo-random functions are highly technical. We shall merely point out a result which achieves a splitting for an averaging operator. To avoid the introduction of too much new material, we shall only use weak almost periodic functions, but a similar result can be reached for pseudo-random functions with the help of a theorem from F. Jakobs.

On an abelian locally compact topological group G, a weak almost periodic function is a continuous function on G, with complex values, such that the set $[T_h f]_{h\in G}$ is a relatively compact subset of $C(G)$ equipped with the weakened topology deduced from the uniform norm. The set $W(G)$ of weak almost periodic functions can be split up into an algebraic direct sum:

$$W(G)=A(G)\oplus F(G)$$

A distinctive aspect of functions belonging to $F(\boldsymbol{R})$ is the following property

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{+T}|f(t)|^2 dt=0$$

(cf. [1] and [2] W. Eberlein).

THEOREM 6. *A stationary operator, continuous for the uniform norm on $W(G)$, is the direct sum of two such operators acting of $A(G)$ and $F(G)$ respectively*:

$$P=P_1\oplus P_2$$

We shall prove theorem 6 in the case where $G=\boldsymbol{R}$ to avoid new notations. Any function f in $A(\boldsymbol{R})$ is the uniform limit of a family of generalized trigonometric polynomials $[g_i]$. But as P is stationary, we get $P(e_\lambda)=a(\lambda)e_\lambda$, so that $P(g_i)$ is also a generalized trigonometric polynomial, which converges towards $P(f)$. This function $P(f)$ is then an element of $A(\boldsymbol{R})$.

Now, let us define $F(x,\lambda,T)=\frac{1}{2T}\int_{-T}^{+T} f(x+t)e^{-i\lambda t}dt$. For the pointwise convergence in x, we get

$$\lim_{T\to+\infty} F(x,\lambda,T)=c_\lambda(f)e^{i\lambda x}$$

where $c_\lambda(f)$ is the generalized Fourier coefficient of f

$$c_\lambda(f)=\operatorname*{Lim}_{T\to+\infty}\frac{1}{2T}\int_{-T}^{+T}f(t)e^{-i\lambda t}\,dt$$

In fact, we have the following inequality

$$|F(x,\lambda,T)-c_\lambda(f)e^{i\lambda x}|\leq|c_\lambda(f)-\frac{1}{2T}\int_{-T}^{+T}f(t)e^{-i\lambda t}\,dt|+\frac{|x|}{T}\|f\|_\infty$$

Let us now use $T(f)$ to denote the set $\bigcup_{h\in R}T_h(f)$ of all translates of f. Using the Riemann converging to $F(x,\lambda,T)$, we see that $F(x,\lambda,T)$ is the uniform limit of elements belonging to the convex and circled hull of $T(f)$. The closure of the circled hull of $T(f)$ in the topology $\sigma(C(\boldsymbol{R}), C'(\boldsymbol{R}))$ is compact because, by definition of a weak almost periodic function, the set $T(f)$ is relatively compact for the weakened topology. Furthermore, by a theorem of Krein-Smulian (cf. [1] W.F. Eberlein) the closed convex hull is also compact. We then have proved that $F(x,\lambda,T)$ belongs to the closed convex and circled hull of Tf. which is a compact set of $\sigma(C(\boldsymbol{R}), C'(\boldsymbol{R}))$. Using a theorem of [1] W.F. Eberlein, pointwise convergence and convergence for the weakened topology are the same on such a compact set. Therefore $C_\lambda(f)e^{i\lambda x}$ is the limit of $F(x,T,\lambda)$ for $\sigma(C(\boldsymbol{R}), C'(\boldsymbol{R}))$ when T tends to infinity. But a linear operator, continuous for the uniform norm on $C(\boldsymbol{R})$ is also continuous for the weakened topology (cf. [1] N. Dunford, J.T. Schwartz). This gives us the proof that $P(F(x,\lambda,T))$, where P acts on $F(x,\lambda,T)$ as a function of x, converges for the weakened topology towards $C_\lambda(f)P(e^{i\lambda x})=c_\lambda(f)a(\lambda)e^{i\lambda x}$. On the other hand, $P(F(x,\lambda,T))=\frac{1}{2T}\int_{-T}^{+T}Pf(x+t)e^{-i\lambda t}\,dt$ as P is continuous for the uniform norm, and stationary. This last expression converges towards $c_\lambda(Pf)e^{i\lambda x}$(for the weakened topology). Finally, we get

$$c_\lambda(Pf)=a(\lambda)c_\lambda(f)$$

But if f is in $F(\boldsymbol{R})$, then $c_\lambda(f)=0$ for all λ in $\boldsymbol{R}$ because of the relation

$$|c_\lambda(f)|\leq\operatorname*{Lim}_{T\to\infty}\left(\frac{1}{2T}\int_{-T}^{+T}|f(t)|^2dt\right)=0$$

Therefore, $c_\lambda(Pf)=0$ for such a function and so Pf is not in $A(\boldsymbol{R})$ but belongs to $F(\boldsymbol{R})$. This ends the proof of theorem 6.

NOTE 1. Theorem 6, added to known results on the algebra $A(G)$, means that it simply remains to study the action of an operator D or $D(0)$ on $F(\bar{G})$. But this "simply" must not be confused with "easily" as there does not exist, to my knowledge, a general result for the spectral synthesis of weak almost periodic functions. Such a result could be derived by Krein-Milman theorem if it were possible to identify, with exponentials e_λ, the extremal points of the closed and circled convex hull of Tf.

NOTE 2. When P is a semi-multiplicative linear operator on $W(G)$, and a stationary bounded operator, then $a(\hat{x})$ is the characteristic function of a subgroup $\wedge$ in $\hat{G}$. Is there any restriction on these subgroups? Naturally, we may construct a non-measurable subgroup for the Lebesgue measure on $\boldsymbol{R}$ (by using the axiom of choice and for a certain selection of points in $\boldsymbol{R}/Q$ for example). On the other hand, a measurable proper subgroup of $\boldsymbol{R}^n$ (or of $\boldsymbol{T}^n$) is of measure 0 for the Haar measure on these groups as can be shown by using the Haar property of the measure and order properties of $\boldsymbol{R}^n$

NOTE 3. Here, we have not studied limits of semi-groups of Reynolds operators. Such a study will sometimes, although not always, lead to theorems on algebras $C(X)$, analogous to martingale theorems on spaces $L^p(\Omega, \mathscr{F}, \mu)$.

NOTE 4. It might also be of practical interest to look for the following simplified prediction problem raised by Professor Nguyen Dinh Ngoc. Let f be a pseudo-random function and P be a given averaging operator(of type $D(\alpha)$ for example). Suppose that we have a detailed knowledge of the values of f on the negative part of the real axis, how is it possible to derive the value of the average Pf at a positive value of the variable?

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