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NOTE ON COMPACT HYPERSURFACES IN A UNIT SPHERE S^{2n+1}

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§ 0. Introduction

Recently, K. Yano and M. Okumura [5] have defined the concept of an (f, g, u, v, λ)-structure in an even-dimensional Riemannian manifold. Hypersurfaces in an almost contact metric manifold or submanifolds of codimension 2 in an almost Hermitian manifold admit an (f, g, u, v, λ)-structure (cf. [5] etc.).

H. Suzuki [4] studied the integrability conditions of this structure.

In terms of this structure, D.E. Blair, G.D. Ludden and K. Yano [1], and M. Nakagawa and I. Yokote [3] have proved

THEOREM 0.1. *If M^{2n} is a complete orientable submanifold with constant scalar curvature satisfying $Kf+fK=0$ and $\lambda\neq$constant, where K denotes the second fundamental tensor on M^{2n}, then M^{2n} is a natural sphere S^{2n} or $S^n\times S^n$.*

In the present paper we investigate the necessary and sufficient condition of antinormal (f, g, u, v, λ)-structure in a Sasakian manifold and study compact hypersurfaces with antinormal (f, g, u, v, λ)-structure in a unit sphere.

§ 1. Preliminaries

We consider a C^∞ differentiable manifold M with an (f, g, u, v, λ)-structure, that is, a Riemannian manifold with metric tensor g which admits a tensor field f of type (1,1), two 1-forms u and v (or two vector fields associated with them), and a function λ satisfying

$$(1.1)\qquad \begin{aligned} &f_j{}^t f_t{}^h=-\delta_j^h+u_ju^h+v_jv^h, \qquad f_j{}^t f_i{}^s g_{ts}=g_{ji}-u_ju_i-v_jv_i,\\ &u_tf_i{}^t=\lambda v_i \text{ or } f_i{}^h u^i=-\lambda v^h, \qquad v_tf_i{}^t=-\lambda u_i \text{ or } f_i{}^h v^i=\lambda u^h,\\ &u_iu^i=v_iv^i=1-\lambda^2, \qquad u_iv^i=0. \end{aligned}$$

Such an M is even-dimensional ([5]).

We put

(1.2) $\quad [f,f]_{ji}=f_j{}^t\nabla_t u_i-f_i{}^t\nabla_t u_j-(\nabla_j f_i{}^t-\nabla_i f_j{}^t)u_t+\lambda(\nabla_j v_i-\nabla_i v_j),$

∇_j denoting the operator of covariant differentiation with respect to the Riemannian connection. If the tensor $[f,f]_{ji}$ vanishes, the (f,g,u,v,λ)-structure is said to be *antinormal* (cf. [4]).

§2. Hypersurfaces in a Sasakian manifold

Let M be an orientable hypersurface of a Sasakian manifold $\widetilde{M}^{2n+1}$. Then there is an (f, g, u, v, λ)-structure induced in M, which has the following properties;

(2.1) $\quad \nabla_j f_i{}^h=-g_{ji}u^h+\delta_j{}^h u_i-k_{ji}v^h+k_j{}^h v_i,$

(2.2) $\quad \nabla_j u_i=f_{ji}-\lambda k_{ji},$

(2.3) $\quad \nabla_j v_i=-k_{jt}f_i{}^t+\lambda g_{ji},$

(2.4) $\quad \nabla_j\lambda=k_{jt}u^t-v_j,$

where k_{ji} is the component of second fundamental tensor of the hypersurface M relative to $\widetilde{M}^{2n+1}$ ([1], [5]).

Since (2.2) implies that $\{x\in M;\ \lambda^2(x)=1\}$ is bordered set, we may only consider $1-\lambda^2\neq 0$ on M.

Substituting (2.1)~(2.4) into (1.2), we find

(2.5) $\quad [f,f]_{ji}=(\nabla_i\lambda)v_j-(\nabla_j\lambda)v_i.$

Thus, we have

LEMMA 2.1. *Let M be an orientable hypersurface of a Sasakian manifold. In order that the induced (f, g, u, v, λ)-structure be antinormal it is necessary and sufficient that it satisfies $\nabla_j\lambda=Av_j$, A being certain differentiable function on M.*

We now assume that

(2.6) $\quad \nabla_j\lambda=Av_j,$

A being non-zero differentiable function on M. Then we have from (2.4)

(2.7) $\quad k_{jt}u^t=(A+1)v_j\,.$

Differentiating (2.6) covariantly and using (2.3), we find

$$\nabla_k\nabla_j\lambda=(\nabla_k A)v_j-A(-k_{kt}f_j{}^t+\lambda g_{kj}),$$

from which,

(2.8) $(\nabla_k A)v_j-(\nabla_j A)v_k+A(k_{jt}f_k{}^t-k_{kt}f_j{}^t)=0.$

Transvecting (2.8) with v^j and using (1.1) and (2.7), we get

(2.9) $(1-\lambda^2)\nabla_k A=(v^t\nabla_t A)v_k+\lambda A(A+1)v_k-Ak_{st}v^s f_k{}^t.$

Substituting (2.9) into (2.8), we obtain

(2.10) $(1-\lambda^2)(k_{jt}f_k{}^t-k_{kt}f_j{}^t)=k_{st}v^s(v_jf_k{}^t-v_kf_j{}^t)$

because of $A\neq 0$.

Transvecting (2.10) with $f_i{}^k$ and using (2.7), we find

$$(1-\lambda^2)\{-k_{ji}-(k_{jt}v^t)v_i-k_{st}f_i{}^sf_j{}^t\}=-(k_{it}v^t)v_j+(k_{st}v^sv^t)v_jv_i+\lambda k_{st}v^sf_j{}^tu_i,$$

from which, taking the skew-symmetric part,

$$(1-\lambda^2)\{(k_{jt}v^t)v_i-(k_{it}v^t)v_j\}=-(k_{it}v^t)v_j+(k_{jt}v^t)v_i+\lambda k_{st}v^s(f_j{}^tu_i-f_i{}^tu_j).$$

Transvecting this with v^i and using $\lambda\neq 0$ and (2.7), we find

(2.11) $k_{jt}v^t=(A+1)u_j+Bv_j,$

where $B=(k_{st}v^sv^t)/(1-\lambda^2)$.

Substituting (2.11) into (2.9) and (2.10), we have respectively

(2.12) $\nabla_j A=Cu_j+Dv_j,$

(2.13) $(1-\lambda^2)(k_{jt}f_k{}^t-k_{kt}f_j{}^t)=\lambda B(u_jv_k-u_kv_j),$

where, we have put

(2.14) $$C=\frac{\lambda AB}{1-\lambda^2},\qquad D=\frac{v^t\nabla_t A}{1-\lambda^2}.$$

Differentiating (2.12) covariantly and taking account of (2.2) and (2.3), we find

$$\nabla_k\nabla_j A=(\nabla_k C)u_j+(\nabla_k D)v_j+C(f_{kj}-\lambda k_{kj})+D(-k_{kt}f_j{}^t+\lambda g_{kj}),$$

from which, substituting (2.13),

(2.15) $$(\nabla_k C)u_j-(\nabla_j C)u_k+(\nabla_k D)v_j-(\nabla_j D)v_k+2Cf_{kj}+\frac{\lambda B}{1-\lambda^2}(u_jv_k-u_kv_j)=0.$$

Transvecting (2.15) with u^kv^j and f^{kj}, we have respectively

$$-v^t\nabla_t C+u^t\nabla_t D-\lambda B-2C\lambda=0,$$

$$\lambda(v^t\nabla_t C-u^t\nabla_t D+\lambda B)+2C(n-1+\lambda^2)=0.$$

Thus, last two equations imply $C=0$ and consequently $\lambda AB=0$. So we have $B=0$ because of $A\neq 0$ and (2.6). Therefore (2.11), (2.12) and (2.13) become respectively

(2.16) $\nabla_j A = D v_j,$

(2.17) $k_{jt} v^t = (A+1) u_j,$

(2.18) $k_{jt} f_k{}^t - k_{kt} f_j{}^t = 0.$

Conversely, if (2.18) satisfied on M, by transvecting $f_i{}^k$, we find

$$k_{jt}(-\delta_i^t + u_i u^t + v_i v^t) - k_{st} f_i{}^s f_j{}^t = 0.$$

Taking the skew-symmetric part of this equation, we have

(2.19) $(k_{jt} u^t) u_i - (k_{it} u^t) u_j + (k_{jt} v^t) v_i - (k_{it} v^t) v_j = 0.$

Transvecting (2.19) with u^i and putting $\overline{A}(1-\lambda^2) = k_{ts} u^t u^s$, $\overline{B}(1-\lambda^2) = k_{ts} u^t v^s$, we get

(2.20) $k_{jt} u^t = \overline{A} u_j + \overline{B} v_j.$

Differentiating (2.20) covariantly and using (2.2) and (2.3), we find

$$(\nabla_k k_{jt}) u^t + k_{jt}(f_k{}^t - \lambda k_k{}^t) = (\nabla_k \overline{A}) u_j + (\nabla_k \overline{B}) v_j + \overline{A}(f_{kj} - \lambda k_{kj}) + \overline{B}(-k_{kt} f_j{}^t + \lambda g_{kj}),$$

from which, taking the skew-symmetric part with respect to k and j,

(2.21) $(\nabla_k k_{jt} - \nabla_j k_{kt}) u^t = (\nabla_k \overline{A}) u_j - (\nabla_j \overline{A}) u_k + (\nabla_k \overline{B}) u_j - (\nabla_j \overline{B}) v_k + 2\overline{A} f_{kj}$

because of (2.18).

On the other hand, differentiating (2.4) covariantly and substituting (2.3), we get

$$\nabla_k \nabla_j \lambda = (\nabla_k k_{jt}) u^t - (\lambda g_{kj} - k_{kt} f_j{}^t),$$

from which, using (2.18),

$$(\nabla_k k_{jt} - \nabla_j k_{kt}) u^t = 0.$$

Thus, (2.21) becomes

(2.22) $(\nabla_k \overline{A}) u_j - (\nabla_j \overline{A}) u_k + (\nabla_k \overline{B}) v_j - (\nabla_j \overline{B}) v_k + 2\overline{A} f_{kj} = 0.$

Transvecting (2.22) with $u^j v^k$ and f^{kj}, we have respectively

$$(1-\lambda^2)(v^t \nabla_t \overline{A} - u^t \nabla_t \overline{B}) + 2\overline{A}\lambda(1-\lambda^2) = 0$$

and

$$\lambda(v^t \nabla_t \overline{A} - u^t \nabla_t \overline{B}) + \overline{A}\{2n - 2(1-\lambda^2)\} = 0,$$

from which, $\overline{A} = 0$. Consequently (2.4) and (2.20) imply that $\nabla_j \lambda = (\overline{B}+1) v_j$.

Hence, using Lemma 2.1, we have

THEOREM 2.2. *Let M be an orientable hypersurface of a Sasakian manifold such that the function λ is not constant. In order that the induced (f, g, u, v, λ)-structure be antinormal it is necessary and sufficient that $Kf+fK=0$, where K is the second fundamental tensor of M with respect to Sasakian manifold.*

§ 3. Compact hypersurface in a unit sphere

Let M be a hypersurface immersed in a unit sphere $S^{2n+1}(1)$ with canonical almost contact structure. Then there is an (f, g, u, v, λ)-structure induced in M, which satisfies (2.1)~(2.4).

We now denote by $R_{kji}{}^h$, R_{ji} and R components of the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of M. The equation of Gauss for the hypersurface M is written as

$$R_{kji}{}^h=\delta_k^h g_{ji}-\delta_j^h g_{ki}+k_k{}^h k_{ji}-k_j{}^h k_{ki}, \tag{3.1}$$

and the equation of Codazzi is given by

$$\nabla_k k_{ji}-\nabla_j k_{ki}=0. \tag{3.2}$$

From (3.1) it follows easily that

$$R_{ji}=(2n-1)g_{ji}+k_t{}^t k_{ji}-k_{jt}k_i{}^t, \tag{3.3}$$

$$R=2n(2n-1)+(k_t{}^t)^2-k_{st}k^{st}. \tag{3.4}$$

We prove the following (cf. [3])

THEOREM 3.1. *Let M be a compact hypersurface with antinormal (f, g, u, v, λ)-structure in a unit sphere $S^{2n+1}(1)$. If λ is not constant, then M is congruent to $S^{2n}(1)$ or $S^n\left(\frac{1}{\sqrt{2}}\right)\times S^n\left(\frac{1}{\sqrt{2}}\right)$ imbedded naturally in $S^{2n+1}(1)$.*

PROOF. Since M has antinormal (f, g, u, v, λ)-structure, (2.7), (2.16), (2.17) and (2.18) are valid on M. From (2.18) we can easily prove that

$$k_t{}^t=0. \tag{3.5}$$

Differentiating (2.17) covariantly, we find

$$(\nabla_k k_{jt})v^t+k_{jt}\nabla_k v^t=(\nabla_k A)u_j+(A+1)\nabla_k u_j\,,$$

from which, using (2.2), (2.3), (2.18) and (3.2),

$$2k_k{}^t k_{ts} f_j{}^s=(\nabla_k A)u_j-(\nabla_j A)u_k+2(A+1)f_{kj}\,. \tag{3.6}$$

Transvecting (3.6) with u^j and taking account of (2.7), (2.16) and (2.17), we have

(3.7) $$(1-\lambda^2)\nabla_k A=-2\lambda(A+1)(A+2)v_k.$$

Substituting (3.7) into (3.6), we get

$$(1-\lambda^2)k_k{}^t k_{ts} f_j{}^s=\lambda(A+1)(A+2)(v_j u_k-v_k u_j)+(1-\lambda^2)(A+1)f_{kj},$$

from which, transvecting f^{kj} and using (2.7) and (2.17),

(3.8) $$k_{st}k^{st}=2(A+1)(A+2-n).$$

Since M is compact, from (3.2), (3.5), (3.6) and (3.7), we can prove that $(A+1)(A+2)=0$ (See [2], [3]). Thus, (3.5) and (3.8) imply that R is constant. Taking account of Theorem 0.1, Theorem 3.1 is proved.

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