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# A NOTE ON $k$-SEMISTRATIFIABLE SPACES

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## 1. Introduction

In [1], D. J. Lutzer introduced $k$-semistratifiable spaces which lies between the class of stratifiable spaces and the class of semi-stratifiable spaces. And he proved that a semimetrizable spaces is stratifiable if and only if it is $k$-semistratifiable.

In this paper, it is shown that (a) the union of two closed $k$-semistratifiable spaces is $k$-semistratifiable, (b) a strong Fréchet $k$-semistratifiable space is stratifiable, (c) the image of a $k$-semistratifiable space under a pseudo-open $k$-mapping is $k$-semistratifiable, (d) the image of a $k$-semistratifiable space under a pseudo-open compact mapping is semi-stratifiable, (e) a regular strong Fréchet space which has a $\sigma$-closure preserving $cs$-network is stratifiable.

Most terms which are not defined in this paper are used in Dugundji [4], all spaces are $T_1$, a mapping is a continuous surjection, and the set of natural numbers is donoted by $N$.

## 2. Definitions and elementary properties

Following definitions are well known (cf. [1], [2]).

DEFINITION 2.1. [1]. A topological space $X$ is a *stratifiable* space if, to each open set $U\subset X$, one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of open subsets of $X$ such that

(a) $\overline{U}_n\subset U$

(b) $\bigcup_{n=1}^{\infty} U_n=U$

(c) $U_n\subset V_n$ whenever $U\subset V$.

This correspondence $U\to\{U_n\}_{n=1}^{\infty}$ is called a *stratification* for the space $X$.

DEFINITION 2.2. [1]. A topological space $X$ is a *semi-stratifiable* space if, to each open set $U\subset X$, one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of closed subsets of $X$ such that

(a) $\bigcup_{n=1}^{\infty} U_n=U$

(b) $U_n\subset V_n$ whenever $U\subset V$.

This correspondence $U \rightarrow \{U_n\}_{n=1}^{\infty}$ is called a *semi-stratification* for the space $X$.

DEFINITION 2.3. [1]. A *k-semistratification* of the space $X$ is a semi-stratification $U \rightarrow \{U_n\}_{n=1}^{\infty}$ for the space $X$ such that given any compact subset $K$ with $K \subset U$, there is a natural number $n$ with $K \subset U_n$.

A space is *k-semistratifiable* if and only if there exists a $k$-semistratification for the space.

D. J. Lutzer showed, in [1], that stratifiable spaces are $k$-semistratifiable and $k$-semistratifiable spaces are semi-stratifiable, but these implications cannot be reversed.

M. Henry, in [2], obtained the following

LEMMA 2.4. *A space $X$ is stratifiable (semi-stratifiable) if and only if to each closed subset $F \subset X$ one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of open subsets of $X$ such that*

(*a*) $F \subset U_n$ *for each* $n$

(*b*) $\bigcap_{n=1}^{\infty} \bar{U}_n = F$ $\quad (\bigcap_{n=1}^{\infty} U_n = F)$

(*c*) $U_n \subset V_n$ *whenever* $U \subset V$.

A correspondence $F \rightarrow \{U_n\}_{n=1}^{\infty}$ is a *dual stratification* (*semi-stratification*) for the space $X$ whenever it satisfies the three conditions of Lemma 2.4.

We may state the following Lemma 2.5. for similar reasons.

LEMMA 2.5. *A space $X$ is k-semistratifiable if and only if to each closed set $F \subset X$, one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of open subsets of $X$ such that*

(*a*) $\bigcap_{n=1}^{\infty} U_n = F$

(*b*) $U_n \subset V_n$ *whenever* $U \subset V$

(*c*) *if* $F \cap K = \phi$ *with $K$ compact in $X$, then there is an open set $U_n$ with* $U_n \cap K = \phi$.

A correspondence $F \rightarrow \{U_n\}_{n=1}^{\infty}$ is a *dual k-semistratification* for the space $X$ whenever it satisfies the conditions of Lemma 2.5.

Certainly, we may suppose that any stratification (semi-stratification, $k$-semistratification) $U \to \{U_n\}$ of $X$ is *increasing*, i.e. $U_u \subset U_{n+1}$ for each $n \in N$, therefore any dual stratification (dual semi-stratification, dual $k$-semistratification) $F \to \{U_n\}$ of $X$ is *decreasing*, i.e. $U_n \supset U_{n+1}$ for each $n \in N$.

LEMMA 2.6. [1]. *Suppose $X$ has a semi-stratification $U \to \{U_n\}_{n=1}^{\infty}$ with the property that if $U$ is open $X$ and $p \in U$, then $p \in \mathrm{Int}(U_n)$ for some $n \in N$. Then $X$ is stratifiable.*

## 3. Main theorems

For this section, we consider the following terminologies. A mapping $f: X \to Y$ is *pseudo-open* [2] if for each $y \in Y$ and any neighborhood $U$ of $f^{-1}(y)$, it follows that $y \in \mathrm{Int}[f(U)]$. A mapping $f: X \to Y$ is *compact* [2] if $f^{-1}(y)$ is compact for each $y \in Y$, and $f$ is a *k-mapping* if $f^{-1}(K)$ is a compact set in $X$ whenever $K$ is a compact set in $Y$.

A *k-network* in a space [3] is a collection of subsets $\mathscr{F}$ such that given any compact subset $K$ and any open set $U$ containing $K$, there is a $F \in \mathscr{F}$ such that $K \subset F \subset U$. A *cs-network* [3] is a collection of subsets $\mathscr{F}$ such that given any convergent sequence $x_n \to x$ and any open set $U$ containing $x$, there is an $F \in \mathscr{F}$ and a positive integer $m$ such that $\{x\} \cup \{x_n : n \geq m\} \subset F \subset U$. Note that any $k$-network is a $cs$-network.

A space $X$ is *strong Fréchet* [5] if whenever $\{A_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets in $X$ and $x$ is a point which is in the closure of each $A_n$, then for each $n \in N$ there exists an $x_n \in A_n$ such that the sequence $x_n \to x$. Clearly, any first countable space is strong Fréchet.

THEOREM 3.1. *If $X, Y$ are closed (in the union) $k$-semistratifiable spaces, then $X \cup Y$ is $k$-semistratifiable.*

PROOF. Let $U$ be open in $X \cup Y$. Then $U = (X \cap U) \cup (Y \cap U)$, and $X \cap U$, $Y \cap U$ is open in $X, Y$ respectively. Set $U_n = (X \cap U)_n \cup (Y \cap U)_n$, for each $n \in N$, where $X \cap U \to (X \cap U)_n$, $Y \cap U \to (Y \cap U)_n$ is an increasing $k$-semistratification for the space $X, Y$ respectively. Then it is easily shown that the correspondence $U \to \{U_n\}_{n=1}^{\infty}$ is a $k$-semistratification for $X \cup Y$.

THEOREM 3.2. *A strong Fréchet $k$-semistratifiable space is stratifiable*

PROOF. Let $U$ be an open set in strong Fréchet $k$-semistratifiable space $X$ and let

$U \to \{U_n\}_{n=1}^{\infty}$ is an increasing $k$-semistratification for the space $X$, and $p \in U$. Assume that $p \in X - \mathrm{Int}(U_n) = \overline{X - U_n}$ for each $n \in N$. Since $X$ is strong Fréchet, there exists an $x_n \in X - U_n$ such that the sequence $x_n \to p$. Furthermore, we may assume that each point $x_n$ is in the open set $U$. Thus $\{x_n : n \in N\} \cup \{p\}$ is a compact subset of $U$. Therefore, there exists a positive integer $m$ such that $\{x_n : n \in N\} \cup \{p\} \subset U_n$ for each $n \geq m$, which is contradict to choicing $x_n$. Thus, by Lemma 2. 6., $X$ is stratifiable.

Using an analogue to proof of Theorem 2.3. in [2], the following Theorem 3.3. and 3.4. may be proved.

THEOREM 3.3. *If $X$ is $k$-semistratifiable and $f$: $X \to Y$ is a pseudo-open $k$-mapping, then $Y$ is $k$-semistratifiable.*

PROOF. If $F \subset Y$ be a closed, then $f^{-1}(F)$ is closed in $X$. For each closed set $F$ of $Y$ and each natural number $n$, let $F_n = \mathrm{Int}\ [f(f^{-1}(F)_n)]$, where $f^{-1}(F) \to f^{-1}(F)_n$ is a dual $k$-semistratification for $X$. We will show that the correspondence $F \to \{F_n\}$ is a dual $k$-semistratification for $Y$. Since $f^{-1}(F) \subset f^{-1}(F)_n$ for each $n \in N$, $f^{-1}(F)_n$ is an open neighborhood of $f^{-1}(y)$ for each $y \in F$, and $f$ is a pseudo-open mapping, therefore, we have $F \subset \bigcap_{n=1}^{\infty} \mathrm{Int}\,[f(f^{-1}(F)_n)] = \bigcap_{n=1}^{\infty} F_n$. For the reverse direction, assume $z \notin F$. Then $f^{-1}(z) \cap f^{-1}(F) = \phi$ with $f^{-1}(z)$ compact in $X$, and therefore there exists a natural number $n$ such that $f^{-1}(z) \cap f^{-1}(F)_n = \phi$. Then $z \notin F_n$ for some $n$. Consequently, we have $F = \bigcap_{n=1}^{\infty} F_n$. Next, if $F$ and $G$ are closed subsets of $Y$ such that $F \subset G$, then clearly $\mathrm{Int}\ [f(f^{-1}(F)_n)] \subset \mathrm{Int}\,[f(f^{-1}(G)_n)]$. Finally, let $K \cap F = \phi$ in $Y$ with $K$ compact and $F$ closed in $Y$. Then $f^{-1}(K) \cap f^{-1}(F) = \phi$, $f^{-1}(K)$ is compact and $f^{-1}(F)$ is closed in $X$. Hence, $f^{-1}(K) \cap f^{-1}(F)_n = \phi$ for some $n$. Therefore, $K \cap \mathrm{Int}\ [f(f^{-1}(F)_n)] = \phi$. By Lemma 2.5., $Y$ is $k$-semistratifiable.

By a minor change of the proof of Theorem 3.3., we have the

THEOREM 3.4. *If $f$: $X \to Y$ is a pseudo-open compact mapping and $X$ is $k$-semistratifiable, then $Y$ is semi-stratifiable.*

Therefore, we have the Theorem 2.3. in [2] as Corollary.

COROLLARY 3.5. *The image of a stratifiable space under a pseudo-open compact* *mapping is semi-stratifiable.*

THEOREM 3.6. *A regular strong Fréchet space which has a $\sigma$-closure preserving cs-network is stratifiable*

PROOF. Let $\mathscr{F}=\bigcup_{n=1}^{\infty}\mathscr{F}(n)$ be a $\sigma$-closure preserving *cs*-network for the regular space $X$. By the Theorem 3.2., it is sufficient to prove that the space $X$ is $k$-semistratifiable. And, since $X$ is regular, we may assume that $\mathscr{F}(n)\subset\mathscr{F}(n+1)$ for each $n\in N$, and that $\mathscr{F}$ is a collection of closed subsets of $X$. For an open set $U$ and for each $n\in N$, let $U_n=\bigcup\{F\subset U : F\in\mathscr{F}(n)\}$. Then $U\rightarrow\{U_n\}_{n=1}^{\infty}$ is a $k$-semistratification for $X$. Indeed, suppose $K\subset U$ with $K$ compact and $U$ open in $X$ and $K-U_n\neq\phi$ for each $n\in N$. We choose $x_n\in K-U_n$. Then, since $K$ is compact, $\{x_n\}$ has an accumulation point $x$. Let $A_n=\{x_i : i\geq n\}$ for each n$\in N$, then $A_n$ is a decreasing sequence and $x\in\overline{A}_n$ for each $n\in N$. Therefore there exists a $p_n\in A_n$ such that the sequence $p_n$ converge to $x$. Hence, there exists a positive integer $m$ and $F\in\mathscr{F}$ such that $\{p_n : n\geq m\}\cup\{x\}\subset F\subset U$. Let $F\in\mathscr{F}(n_0)$. Then $\{p_n : n\geq m\}\subset U_i$ for each $i\geq n_0$, which is contradict to choicing $x_n$. The remain part is clear.

With the aid of Theorem 3.1. of [6], we have the following

COROLLARY 3.7. *A regular first countable space which has a $\sigma$-closure preserving cs-network is a Nagata space.*

A regular space which has a $\sigma$-locally finite $k$-network is called an *$\aleph$-space* [1].

COROLLARY 3.8. *A strong Fréchet $\aleph$-space is stratifiable.*

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