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ON C-CONFORMAL KILLING TENSOR IN A COSYMPLECTIC MANIFOLD

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0. Introduction.

It is well known that a skew symmetric tensor u_{bc} is called a conformal Killing tensor if it satisfies the following equation:

$$\nabla_a u_{bc}+\nabla_b u_{ac}=2\rho_c g_{ab} \tag{0.1}$$

where ρ_c is a certain vector field. It is a generalization of conformal Killing vector satisfying the Killing-Yano's equation.

On the other hand, Tachibana [2] has defined a conformal Killing tensor in another way. By the definition, a skew symmetric tensor field u_{bc} called a conformal Killing tensor if there exists a vector field p^a satisfying

$$\nabla_a u_{bc}+\nabla_b u_{ac}=2p_c g_{ab}-p_a g_{bc}-p_b g_{ac} \tag{0.2}$$

Afterward, Yamaguchi [4] has defined a product conformal Killing tensor in a locally product Riemannian manifold and obtained some results. And Chen [1] has defined a F-conformal Killing tensor in Kählerian space and generalized some results.

In this paper we shall define a C-conformal Killing tensor in a cosymplectic manifold and we obtain analogues results to a conformal Killing tensor.

1. Preliminaries.

Let M be a $(2n+1)$ -dimensional differentiable manifold with an almost contact metric structure $(\phi_b^{\ a},\ \xi^a,\ \eta_b,\ g_{ab})$ satisfying

$$\phi_c^{\ a}\phi_b^{\ c}=-\delta_b^{\ a}+\xi^a\eta_b \tag{1.1}$$

$$\phi_b^{\ a}\xi^b=0,\qquad \phi_b^{\ a}\eta_a=0,\qquad \xi^a\eta_a=1 \tag{1.2}$$

$$g_{ab}\xi^b=\eta_a \tag{1.3}$$

$$g_{cd}\phi_a^{\ c}\phi_b^{\ d}=g_{ab}-\eta_a\eta_b \tag{1.4}$$

If the almost contact structure is normal, then the manifold M is called a normal contact manifold or a Sasakian manifold. An almost contact metric structure is said to be cosymplectic if it is normal and 2-form $\phi_{ab}=\phi_a{}^c g_{cb}$ and 1-form η_b are both closed. It is known that the cosymplectic is characterized by

(1.5) $$\nabla_c\phi_b{}^a=0, \qquad \nabla_c\eta_b=0.$$

Let R_{abcd} and R_{ab} be the curvature tensor and the Ricci tensor respectively. In a cosymplectic manifold, by virtue of (1.5) we have

(1.6) $$R_{abcd}\eta^d=0, \qquad R_{ad}\eta^d=0$$

If we put

$$F_{ab}=\frac{1}{2}R_{abcd}\phi^{cd}$$

then making use of the Ricci identity for ϕ_{cd}, we have

$$R_{abc}{}^t\phi_{td}+R_{abd}{}^t\phi_{ct}=0$$

Contracting g^{bc} to the last equation, we obtain

(1.7) $$R_a{}^t\phi_{tb}=F_{ab}$$

from which

(1.8) $$F_a{}^t\phi_{tb}=-R_{ab}$$

2. *C*-conformal Killing tensor.

In this section we shall define a *C*-conformal Killing tensor in a cosymplectic manifold *M*. For a skew symmetric tensor field u_{cd} if there exists a vector field p^a such that

(2.1) $$\nabla_b u_{cd}+\nabla_c u_{bd}=2p_d g_{bc}-p_b g_{cd}-p_c g_{bd}-2p_d\eta_b\eta_c+p_b\eta_c\eta_d +p_c\eta_b\eta_d+3(\bar{p}_b\phi_{cd}+\bar{p}_c\phi_{bd})$$

where we put $\bar{p}_c=\phi_c{}^t p_t$, then we call u_{cd} a *C*-conformal Killing tensor and p^a the associated vector of u_{cd}. The associated vector of u_{cd} is given by

(2.2) $$p_d=\frac{\nabla^c u_{cd}}{2(n+1)}+\frac{(\nabla^b u_{bc})\eta^c\eta_d}{2n(n+1)}$$

and if p_d vanishes identically then u_{cd} is a Killing tensor.

By the definition and (1.4), we have

(2.3) $$\bar{p}_c p^c=0, \qquad \bar{p}_c\eta^c=0$$

(2.4) $$p_c p^c-\bar{p}_c\bar{p}^c=\lambda^2$$

where $\lambda=p_c\eta^c$ is a scalar function.

Since we obtain the following formula for any skew symmetric tensor T_{ab},

$$\nabla^a\nabla^b T_{ab}=0,$$

from (2.2) we get

$$(\nabla_b p_c)\eta^b\eta^c=(n+1)\nabla^c p_c=0. \tag{2.5}$$

Next, we shall seek for differential equations of second order satisfied by u_{cd}. If we put

$$G_{ab}=g_{ab}-\eta_a\eta_b, \tag{2.6}$$

then the equation (2.1) becomes

$$\nabla_b u_{cd}+\nabla_c u_{bd}=2p_d G_{bc}-p_b G_{cd}-p_c G_{bd}+3(\bar{p}_b\phi_{cd}+\bar{p}_c\phi_{bd}) \tag{2.7}$$

Operating ∇_a to the last equation, we get

$$\nabla_a\nabla_b u_{cd}+\nabla_a\nabla_c u_{bd}=2p_{ad}G_{bc}-p_{ab}G_{cd}-p_{ac}G_{bd}+3(\bar{p}_{ab}\,\phi_{cd}+\bar{p}_{ac}\phi_{bd}) \tag{2.8}$$

where we put

$$p_{ab}=\nabla_a p_b, \qquad \bar{p}_{ab}=\nabla_a\bar{p}_b=p_{ac}\phi_b^{\ c}.$$

Changing the indices a, b, c cyclically, adding these two equations and subtracting (2.8), we obtain

$$\begin{aligned}&2\nabla_a\nabla_b u_{cd}-2R_{cba}{}^t u_{dt}-R_{bad}{}^t u_{ct}-R_{acd}{}^t u_{bt}-R_{bcd}{}^t u_{at}\\&=2(p_{ad}G_{bc}+p_{bd}G_{ca}-p_{cd}G_{ab})-(p_{ab}+p_{ba})G_{dc}-(p_{ac}-p_{ca})G_{db}\\&-(p_{bc}-p_{cb})G_{da}+3(\bar{p}_{ab}+\bar{p}_{ba})\phi_{cd}+3(\bar{p}_{ac}-\bar{p}_{ca})\phi_{bd}+3(\bar{p}_{bc}-\bar{p}_{cb})\phi_{ad}\end{aligned} \tag{2.9}$$

Again, changing the indices b, c, d cyclically and adding these three equations, we have

$$\begin{aligned}&2\nabla_a\nabla_b u_{cd}-R_{cba}{}^t u_{dt}-R_{bda}{}^t u_{ct}-R_{dca}{}^t u_{bt}\\&=(p_{bd}-p_{db})G_{ca}+(p_{cb}-p_{bc})G_{ad}+(p_{cd}-p_{dc})G_{ab}+(p_{db}-p_{bd})G_{ba}\\&-2p_{ac}G_{bd}+2p_{ad}G_{bc}+(\bar{p}_{bc}-\bar{p}_{cb})\phi_{ad}+(\bar{p}_{cd}-\bar{p}_{dc})\phi_{ab}+(\bar{p}_{db}-\bar{p}_{bd})\phi_{ac}\\&+2(\bar{p}_{da}-\bar{p}_{ad})\phi_{bc}+2(\bar{p}_{ac}-\bar{p}_{ca})\phi_{bd}+2(\bar{p}_{ab}+\bar{p}_{ba})\phi_{cd},\end{aligned} \tag{2.10}$$

where we have used the following equation

$$\begin{aligned}\nabla_a\nabla_b u_{cd}+\nabla_a\nabla_c u_{db}+\nabla_a\nabla_d u_{bc}=3(&\nabla_a\nabla_b u_{cd}+p_{ac}G_{bd}-p_{ad}G_{bc}+\bar{p}_{ad}\phi_{bd}\\&-\bar{p}_{ac}\phi_{bd}+2\bar{p}_{ad}\phi_{dc}).\end{aligned}$$

3. Integral formula.

In this section we shall prove some integral formula about a tensor field. Let

u_{cd} be a C-conformal Killing tensor. Then we obtain

$$(3.1)\qquad \nabla^a\nabla_a u_{cd} - R_c^{\ a} u_{da} - R^a{}_{cd}{}^b u_{ab}$$
$$= -(2n-3)p_{cd} - p_{dc} - 3\bar{p}_a{}^a\phi_{cd} + 2p_{ad}\eta^a\eta_c + (p_{ca}-p_{ac})\eta^a\eta_d$$
$$-3(\bar{p}_{ac}-\bar{p}_{ca})\phi_d{}^a$$

by transvecting (2.9) with g^{ab}.

Now, we shall show that a skew symmetric tensor u_{cd} satisfying (3.1) is a C-conformal Killing tensor provided that M is compact.

If we put

$$(3.2)\qquad U_{bcd} = \nabla_b u_{cd} + \nabla_c u_{bd} - 2p_d G_{bc} + p_b G_{dc} + p_c G_{db} - 3(\bar{p}_b\phi_{cd} + \bar{p}_c\phi_{bd})$$

for a skew symmetric tensor u_{cd}, where p_c and $\bar{p}_b$ are given by

$$p_c = \frac{\nabla^b u_{bc}}{2(n+1)} + \frac{(\nabla^a u_{ab})\eta^b\eta_c}{2n(n+1)}$$

$$\bar{p}_b = \frac{1}{6(n+1)}(\nabla_b u_{cd} + \nabla_c u_{bd})\phi^{cd}$$

Simple computations give us the following

$$(3.3)\qquad U_{bcd}U^{bcd} = 2U_{bcd}\nabla^b u^{cd}$$

$$(3.4)\qquad u^{cd}U_{bcd} = u^{cd}(\nabla^a\nabla_a u_{cd} - R_c^{\ a} u_{da} - R^a{}_{cd}{}^b u_{ab} + (2n-3)p_{cd} + {}_{dc} - 2p_{ad}\eta^a\eta_c$$
$$-(p_{ca}-p_{ac})\eta^a\eta_d - 3\bar{p}_a{}^a\phi_{cd} + 3(\bar{p}_{ac}-\bar{p}_{ca})\phi_d{}^a).$$

Substituting (3.3) and (3.4) into

$$\nabla^b(U_{bcd}u^{cd}) = \nabla^b U_{bcd}u^{cd} + U_{bcd}\nabla^b u^{cd}.$$

Thus we have

THEOREM 3.1. *In a compact cosymplectic manifold M, the following integral formula is valid for any skew symmetric tensor u_{cd}*

$$\int_M\Big[u^{cd}(\nabla^a\nabla_a u_{cd} - R_c^{\ a} u_{da} - R^a{}_{cd}{}^b u_{ab} + (2n-3)p_{cd} + p_{dc} - 2p_{ad}\eta^a\eta_c$$
$$-(p_{ca}-p_{ac})\eta^a\eta_d - 3\bar{p}_a{}^a\phi_{cd} + 3(\bar{p}_{ac}-\bar{p}_{ca})\phi_d{}^a + \frac{1}{2}U_{bcd}U^{bcd}\Big]d\sigma = 0,$$

where $d\sigma$ means the volume element of M, p_{cd} and $\bar{p}_{cd}$ are given by

$$p_{cd} = \frac{\nabla_c\nabla^b u_{bd}}{2(n+1)} + \frac{(\nabla_c\nabla^b u_{ba})\eta^a\eta_d}{2n(n+1)}$$

$$\bar{p}_{cd}=\frac{1}{6(n+1)}(\nabla_c\nabla_d u_{ab}+\nabla_c\nabla_a u_{db})\phi^{ab}.$$

Thus we have

THEOREM 3.2. *In a compact cosymplectic manifold M, a necessary and sufficient condition for any skew symmetric u_{cd} to be a C-conformal Killing tensor is* (3.1).

4. A manifold of constant C-holomorphic sectional curvature.

It has been shown that in a Sasakian manifold or a cosymplectic manifold of constant C-holomorphic sectional curvature k, the curvature tensor R_{abcd} has the form

$$R_{abcd}=a(g_{ad}g_{bc}-g_{ac}g_{bd})+b(\phi_{ad}\phi_{bc}-\phi_{ac}\phi_{bd}-2\phi_{ab}\phi_{cd} -g_{ad}\eta_b\eta_c-g_{bc}\eta_a\eta_d+g_{ac}\eta_b\eta_d+g_{bd}\eta_a\eta_c) \tag{4.1}$$

where $a=(k+3)/4$ *and* $b=(k-1)/4$ in Sasakian manifold, $a=b=k/4$ in cosymplectic manifold. This formula was shown for the Sasakian case by Ogiue and for the cosymplectic case by Blair.

Now we shall show the following

THEOREM 4.1. *In a cosymplectic manifold of constant C-holomorphic sectional curvature, the covariant derivative $\nabla_c v_d$ of any Killing vector v_d is a C-conformal Killing tensor.*

PROOF. Let v_d be a Killing vector. Then as is well known we have

$$\nabla_b\nabla_c v_d+R_{abcd}v^a=0. \tag{4.2}$$

Substituting (3.1) into the last equation, we get

$$\nabla_b\nabla_c v_d=-c(v_d g_{bc}-v_c g_{bd}-\bar{v}_d\phi_{bc}-\bar{v}_c\phi_{bd}+2\bar{v}_b\phi_{cd} -v_d\eta_b\eta_c-\lambda g_{bc}\eta_d+v_c\eta_b\eta_d+\lambda g_{bd}\eta_c), \tag{4.3}$$

where $\bar{v}_d=\phi_d{}^a v_a$ and $\lambda=v^a\eta_a$.

If we put

$$p_d=-c(v_d-\lambda\eta_d), \qquad \bar{p}_d=\phi_d{}^a p_a=-c\bar{v}_d,$$

then we obtain

$$\nabla_b\nabla_c v_d=p_d g_{bd}-p_c g_{bd}-\bar{p}_c\phi_{bd}-\bar{p}_b\phi_{cd}+2\bar{p}_b\phi_{cd} -p_d\eta_b\eta_c-p_c\eta_b\eta_d.$$

Changing the indices b and c, adding these two equations, we have

$$\nabla_c\nabla_b v_d+\nabla_b\nabla_c v_d=(2p_d g_{bc}-p_b g_{cd}-p_c g_{bd})-(2p_d\eta_b\eta_c -p_b\eta_c\eta_d-p_c\eta_b\eta_d+3(\bar{p}_b\phi_{cd}+\bar{p}_c\phi_{bd})$$

This equation shows that $\nabla_c v_d$ is a C-conformal Killing tensor.

We know the converse of Theorem 3.1 is valid as follows.

THEOREM 4.2. *In a cosymplectic manifold M, if Lie algebra of all Killing vectors v_d is transitive and the covariant derivative $\nabla_c v_d$ of any Killing vector v_d is a C-conformal Killing tensor, then M is a manifold of constant C-holomorphic sectional curvature.*

PROOF. Taking $u_{cd}=\nabla_c u_d$ in (2.7) and by making use of (3.2), we have

$$-(R_{abcd}+R_{acbd})u^a = (2p_d G_{bc}-p_b G_{cd}-p_c G_{bd})+3(p_b\phi_{cd}+p_c\phi_{bd}) \tag{4.4}$$

Transvecting (4.4) with g^{bc} and ϕ^{cd} respectively, we have

$$-R_{ad}u^a=2(n+1)p_d \tag{4.5}$$

$$-F_{ab}u^a=2(n+1)\bar{p}_b \tag{4.6}$$

by virtue of (1.5).

Substituting (4.5) and (4.6) into (4.4), we have

$$(R_{abcd}+R_{acbd})u^a=\frac{1}{2(n+1)}(2R_{ad}G_{bc}-R_{ab}G_{cd}-R_{ac}G_{bd} +3(F_{ab}\phi_{cd}+F_{ac}\phi_{bd})u^a)$$

Since the last equation holds for any vector u^a, we obtain

$$R_{abcd}+R_{acbd}=\frac{1}{2(n+1)}(2R_{ad}G_{bc}-R_{ab}G_{cd}-R_{ac}G_{bd} +3(F_{ab}\phi_{cd}+F_{ac}\phi_{bd})) \tag{4.7}$$

Transvecting (4.7) with g^{ad} and taking account of (1.8), we have

$$R_{bc}=\frac{1}{2n}RG_{bc} \tag{4.8}$$

Substituting the last equation into (4.7), we have

$$R_{abcd}+R_{acbd}=\frac{R}{2n(n+1)}(2G_{ad}G_{bc}-G_{ab}G_{cd}-G_{ac}G_{bd} +3(\phi_{ab}\phi_{cd}+\phi_{ac}\phi_{bd})) \tag{4.9}$$

Interchanging indices b, c, d in (4.9) as $b\to c\to d\to b$ and then substracting what follows from (4.9), we have

$$R_{acbd}=\frac{R}{2n(n+1)}(G_{ad}G_{bc}-G_{ab}G_{cd}+\phi_{ab}\phi_{cd}-\phi_{ac}\phi_{bd}-2\phi_{ac}\phi_{bd})$$

taking account of $G_{bc}=g_{bc}-\eta_b\eta_c$, the last equation becomes

$$R_{acbd}=\frac{R}{2n(n+1)}(g_{ad}g_{bc}-g_{ab}g_{cd}+\phi_{ab}\phi_{cd}-\phi_{ac}\phi_{bd}-2\phi_{ac}\phi_{bd} -g_{ad}\eta_b\eta_c-g_{bc}\eta_a\eta_d+g_{ab}\eta_c\eta_d+g_{cd}\eta_a\eta_b) \tag{4.10}$$

Thus the proof is complete.

Let us assume that $c\neq 0$. If we put

$$q_{cd}=u_{cd}+\frac{1}{c}\nabla_c p_d \tag{4.11}$$

then by virtue of (4.3) and (2.1), it follows that

$$\nabla_b q_{cd}+\nabla_c q_{bd}=0,$$

which means q_{cd} is a Killing tensor. Consequently, a C-conformal tensor u_{cd} is decomposed in the form

$$u_{cd}=q_{cd}+p_{cd} \tag{4.12}$$

where q_{cd} is a Killing tensor and $p_{cd}=-\frac{1}{a}\nabla_c p_d$ is a closed C-conformal Killing tensor. Thus we have

THEOREM 4.3. *In a cosymplectic manifold of constant C-holomorphic sectional curvature $a=R/2n(n+1)\neq 0$, a C-conformal Killing tensor u_{cd} is decomposed in the form*

$$u_{cd}=q_{cd}+p_{cd}$$

where q_{cd} is a Killing tensor and p_{cd} is a closed C-conformal Killing tensor. In this case p_{cd} is the form

$$p_{cd}=-\frac{1}{a}\nabla_c p_d$$

where p_d is the associated vector of u_{cd}. Conversely if q_{cd} is a Killing tensor and p_c is a Killing vector, then u_{cd} given by (4.12) is a C-conformal Killing tensor.

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