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SEMIGROUPS ON THE DISK WITH THREAD BOUNDARIES

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In an earlier work [5] commutative semigroups on the two-cell without zero divisors whose boundary consisted of two usual unit intervals were determined. In this note the semigroups with zero divisors are determined. Moreover, if S is a commutative semigroup on the two-cell whose boundary consists of two threads with $E(S)$ (the set of idempotents of S) $=\{z, i\}$ where z is the zero for S and i is the identity for S, then a classification for S is obtained when the results here are combined with the results in [5]. Standard notation found in [7] will be used here. In particular, we will let $(I, \cdot)$ be the usual unit interval. Also A^* will represent the topological closure of A.

1.1. DEFINITION. A commutative semigroup S is said to have *property* (β) if

(1) S is topologically a two-cell

(2) The boundary of S is the union of two threads

(3) S has zero divisors

(4) $E(S)=\{z, i\}$ where z is a zero for S and i is an identity for S.

Since commutative semigroups on the two-cell without zero divisors and whose boundary consists of two usual unit intervals have been determined in [5], we shall concern ourselves only with semigroups with zero divisors. In section 1 we exibit a method for construction semigroups with property (β); later in section 2 we show that these are the only examples.

1.2. Consider the collection $\mathscr{M}$, $\mathscr{M}=\{M_i, (a_j, b_j)$: $i=0, 1, 2, 3, \cdots, j=1, 2, 3, \cdots\}$ whose elements satisfy the following conditions:

(1) M_i is a closed ideal of $(I, \cdot)\times(I, \cdot)$ for $i=0, 1, 2, 3, \cdots$.

(2) $(\{0\}\times[0, 1])\cup([0, 1]\times\{0\})\subsetneq M_0$.

(3) $a_j, b_j\in(0,1)$ for $j=1, 2, 3, \cdots$.

(4) $M_i\cap M_j\subset M_0$ for $i\neq j$.

(5) If $i=j$, then $M_i\cap\{(a_j^{wt}, b_j^{w-wt}) : 0\leq t\leq 1\}\subset M_0$ for $w\geq 1$ and

$M_i\cap\{(a_j^{wt}, b_j^{w-wt}) : 0\leq t\leq 1\}\cap M_0=\phi$ for $0<w<1$.

1.3. LEMMA. *If M_i, $i=0, 1, 2, 3, \cdots$ are closed ideals of $(I, \cdot)\times(I, \cdot)$ such that $M_i \cap M_j \subset M_0$ for $i \neq j$, then $\bigcup_{i=0}^{\infty} M_i$ is a closed ideal of $(I, \cdot)\times(I, \cdot)$. Moreover, if $\{w_n\}$ is a sequence in $\bigcup_{i=0}^{\infty} M_i$ which converges to w and $\{w_n\}$ is not eventually in any M_k, then $w \in M_0$.*

PROOF. Since the union of ideals is an ideal we need only show $\bigcup_{i=0}^{\infty} M_i$ is closed. We distinguish two cases.

Case 1. Let $w_n \to w$ with $\{w_n\} \subset \bigcup_{i=0}^{m} M_i$ for some natural number m. If $\{w_n\}$ is eventually in M_j for some j, then $w_n \to w \in M_j$, since M_j is closed. If $\{w_n\}$ is not eventually in M_j for some j, then there exist subsequences $\{w_n'\} \subset M_j$ and $\{w_n''\} \subset M_k$ for some j and k, $k \neq j$. Now $w_n' \to w \in M_j$ and $w_n'' \to w \in M_k$. Hence $w \in M_j \cap M_k \subset M_0$, and $w \in \bigcup_{i=0}^{\infty} M_i$.

Case 2. Now suppose $\{w_n\}$ is contained in an infinite number of M_i's. Then there exists a subsequence $\{w_n'\}$ with each w_n' in a different M_k. Let $w_n'=(x_n, y_n)$ and $w=(x,y)$. Since $w_n \to w$, $x_n \to x$ and $y_n \to y$. We claim $x_n \geq x_{n+1}$ implies $y_n < y_{n+1}$, since otherwise $y_n \geq y_{n+1}$, and hence there exist x_n', y_n' such that $(x_n, y_n)\cdot(x_n', y_n')=(x_{n+1}, y_{n+1})$. Hence (x_{n+1}, y_{n+1}), $(x_n, y_n) \in M_k$ for some k, and this is a contradiction. Now there exists a subsequence of $\{x_n\}$ or $\{y_n\}$ which is decreasing, say $\{\bar{x}_n\}$ is decreasing and $\bar{x}_n \to x$. Now $(\bar{x}_n, \bar{y}_n)=\bar{w}_n \to w$ and there exist $\tilde{x}_n, \tilde{y}_n$ such that $(\bar{x}_n, \bar{y}_n)(\tilde{x}_n, 1)=(\bar{x}_{n+1}, \bar{y}_{n+1})\ (1, \tilde{y}_n)$. Hence $(\bar{x}_n, \bar{y}_n)(\tilde{x}_n, 1)$, $(\bar{x}_{n+1}, \bar{y}_{n+1})(1, \tilde{y}_{n+1}) \in M_0$. Now consider the sequence $(\bar{x}_n, \bar{y}_n)(\tilde{x}_n, 1)$. Since $\bar{x}_n \tilde{x}_n = \bar{x}_{n+1}$, and $\bar{x}_n \to x$, $\bar{x}_{n+1} \to x$, we have $(\bar{x}_{n+1}, \bar{y}_n) \to (x, y)$. But $(\bar{x}_n, \bar{y}_n)(\tilde{x}_n, 1) \in M_0$ implies $(x, y) \in M_0$, and we have $w \in \bigcup_{i=0}^{\infty} M_i$.

1.4. DEFINITION. Define a relation $R(\mathscr{M})$ on $(I, \cdot)\times(I, \cdot)$ by $(\hat{r}, \hat{s}) \in R(\mathscr{M})$ if

(1) $\hat{r}=\hat{s}$

(2) $\hat{r}, \hat{s} \in M_0$

(3) $\hat{r}, \hat{s} \in M_i$ and there exist a w with $0<w<1$ such that $\hat{r}$ and $\hat{s}$ are contained in the same component of $(M_i \cap \{(a_i^{wt}, b_i^{w-wt}) : 0 \leq t \leq 1\})$.

1.5. LEMMA. *The relation $R(\mathscr{M})$ is a closed congruence, and hence*

$[(I, \cdot)\times(I, \cdot)]/R(\mathscr{M})$ is a semigroup.

PROOF. The proof follows directly from 1.3 and lemma 1 of [5].

1.6. DEFINITION. We say the collection $\mathscr{M}$ defined in 1.2 is of

Class A if for $M_i \in \mathscr{M}$, $M_i \cap ((0.1] \times \{1\}) \cup (\{1\} \times (0,1]) = \phi$.

Class B if for $M_0 \in \mathscr{M}$, $M_0 \cap (I \times \{1\}) = [0,1/2] \times \{1\}$ and $M_i \in \mathscr{M}$ implies $M_i \cap (\{1\} \times (0,1]) = \phi$.

Class C if for $M_0 \in \mathscr{M}$, $M_0 \cap ((I \times \{1\}) \cup (\{1\} \times I)) = ([0,1/2] \times \{1\}) \cup (\{1\} \times [0, 1/2])$, and $M_i \in \mathscr{M}$ implies $\pi_1(M_i \cap [(1/2,1] \times \{1\}]) \cap \pi_2(M_i \cap [\{1\} \times (1/2,1]]) = \phi$.

Let $\mathscr{M}$ be of class A, B, or C and ϕ: $(I, \cdot)\times(I, \cdot) \to [(I, \cdot)\times(I, \cdot)]/R(\mathscr{M})$ be the natural map. It is easily seen that this map is monotone and no equivalence class of $R(\mathscr{M})$ separates $(I, \cdot)\times(I, \cdot)$. Employing a theorem of Whyburn [8] we see $[(I, \cdot)\times(I, \cdot)]/R(\mathscr{M})$ is a two cell. Since $(I, \cdot)\times(I, \cdot)$ is commutative, we have $(I, \cdot)\times(I, \cdot)/ R(\mathscr{M})$ is commutative. Moreover, $[(I, \cdot)\times(I, \cdot)]/R(\mathscr{M})$ is a commutative semigroup satisfying property (β). Thus we have the following:

1.7. THEOREM. *If $\mathscr{M}$ is a collection satisfying 1.2 and of Class A, B or C, then $[(I, \cdot)\times(I, \cdot)]/R(\mathscr{M})$ is a commutative semigroup with property (β). Furthermore, if $\mathscr{M}$ is of Class A, the boundary of $[(I, \cdot)\times(I, \cdot)]/R(\mathscr{M})$ is the union of two usual unit intervals; if $\mathscr{M}$ is of Class B, the boundary of $[(I, \cdot)\times(I, \cdot)]/R(\mathscr{M})$ is the union of a usual unit interval and a nil thread; if $\mathscr{M}$ is of Class C, then the boundary of $[(I, \cdot)\times(I, \cdot)]/R(\mathscr{M})$ is the union of two nil threads.*

Now we will show that the above semigroups are the only semigroups satisfying property (β). Throughout S will be a commutative semigroup with property (β). We intend to construct a collection $\mathscr{M}$ such that $[(I, \cdot)\times(I, \cdot)]/R(\mathscr{M})$ is isomorphic to S.

2.1. LEMMA (*Hilderbrant* [6]) *If the boundary of $S = U \cup V$ where U and V are threads, then $U \cap V = \{z, i\}$ and $S = U \cdot V$.*

Since U and V are threads with $E(S) = \{z, i\}$, there exist homomorphisms f: $(I, \cdot) \to U$ and g: $(I, \cdot) \to V$.

2.2. LEMMA. *There exist a continuous homomorphism h: $(I, \cdot)\times(I, \cdot) \to S$ which is onto S.*

PROOF. Define h: $(I, \cdot)\times(I, \cdot) \to S$ by $h(x,y) = f(x)\cdot g(y)$. Since $S = U \cdot V$, h is the product of homomorphisms, and h is continuous; h: $(I, \cdot)\times(I, \cdot) \to S$ is a continuous homomorphism onto S.

Moreover, the homomorphism h: $(I, \cdot)\times(I, \cdot)\to S$ onto S can be chosen such that

(a) if U and V are usual unit intervals then the restriction of h to $\{1\}\times I$ and $I\times\{1\}$ are iseomorphisms onto U and V respectively.

(b) if U is a usual unit interval and V is a nil thread, then h restricted to $I\times\{1\}$ is an iseomorphism onto U while h maps $\{1\}\times[0, 1/2]$ onto z and is 1-1 restricted to $\{1\}\times[1/2, 1]$.

(c) if U and V are nil threads, then h maps $\{1\}\times[0, 1/2]$ and $[0, 1/2]\times\{1\}$ to z and is 1-1 restricted to $\{1\}\times[1/2, 1]$ and $[1/2, 1]\times\{1\}$.

The homomorphism h will always be chosen this way.

Combining the results in [1], [3], and [5] it can be shown that for each $s\in S-\{z\}$ there exists $a_s, b_s\in(0, 1)$ and $t_s, t_s'\in[0, 1]$ such that $h^{-1}(s)=\{(a_s^t, b_s^{1-t})\colon t_s\leq t\leq t_s'\}$. Since S has this property we will call S rooted.

2.3. NOTATION. Let $J=\{s\colon s\in S$ and $h^{-1}(s)$ is not a point$\}$. Note J is an ideal of S.

2.4 DEFINITION. Let P be an ideal of S. We say P is *positive* if for $p, p'\in P-\{z\}$ there exist $s, s'\in S$ such that $ps=p's'\neq z$.

2.5 LEMMA. *If S is rooted and $P\subset J$ is a positive ideal, and $p, p'\in P-\{z\}$ with $h^{-1}(p)=\{(a^t, b^{1-t})\colon t_1\leq t\leq t_2,\ t_1\neq t_2\}$, then there exists $w\in(0, \infty)$ such that $h^{-1}(p')=\{(a^{wt}, b^{w(1-t)})\colon t_1'\leq t\leq t_2',\ t_1'\neq t_2'\}$.*

PROOF. We know there exist $c, d\in(0, 1)$ such that $h^{-1}(p')=\{(c^t, d^{1-t})\colon t_1'\leq t\leq t_2',\ t_1'\neq t_2'\}$. Now there exist $s, s'\in S$ such that $sp=s'p'\neq z$ and $sp, s'p'\in J$. Let $h(x_1, y_1)=s$ and $h(x_2, y_2)=s'$. For $(x, y)\in h^{-1}(p)\cdot(x_1, y_1)$ we have $h(x, y)=ps$, and for $(x', y')\in h^{-1}(p')\cdot(x_2, y_2)$ we have $h(x', y')=p's'$. But lemma 1 [5] implies $h^{-1}(p)\cdot(x_1, y_1)=\{(a^{u\delta}, b^{u(1-\delta)})\colon \delta_1\leq\delta\leq\delta_2,\ \delta_1\neq\delta_2\}$ and $h^{-1}(p')\cdot(x_2, y_2)=\{(c^{\gamma n}, d^{\gamma(1-n)})\colon n_1\leq n\leq n_2,\ n_1\neq n_2\}$. However, there exists $m, q\in(0, 1)$ such that $h^{-1}(sp)=h^{-1}(s'p')=\{(m^\lambda, q^{1-\lambda})\colon \lambda_1\leq\lambda\leq\lambda_2,\ \lambda_1\neq\lambda_2\}$. This implies $a^u=m=c^\gamma$ and $b^u=q=d^\gamma$, or $c=a^{u/\gamma}$ and $d=b^{u/\gamma}$. This completes the proof.

2.6. DEFINITION. Let P be a positive ideal of S with $P\subset J$. If $p\in P-\{z\}$ with $h^{-1}(p)\supset\{(a^t, b^{1-t})\colon t_1\leq w\leq t\leq t_2,\ t_1\neq t_2\}$, if also N is an ideal of $(I, \cdot)\times(I, \cdot)$ such that $h^{-1}(P-\{z\})\subset N$ and $N\subset h^{-1}(P)$, then we define $Com((P,N);(a^w,b^{1-w}))$ to be the component of $(N\cap\{(a^t,b^{1-t}): 0\leq t\leq 1\})$ containing (a^w,b^{1-w}).

2.7 LEMMA. *If S is rooted and* $P\subset J$ *is a positive ideal and N is an ideal of* $(I,\cdot)\times(I,\cdot)$ *such that* $h^{-1}(P-\{z\})\subset N\subset h^{-1}(P)$ *and* $p\in P-\{z\}$ *with* $h^{-1}(p)=\{(a^t,b^{1-t}): t_1\leq t\leq t_2,\ t_1\neq t_2\}$, *then* $h^{-1}(p)=Com((P,N);\ (a^{t_1},b^{1-t_1}))$.

PROOF. Let $h^{-1}(p)=\{(a^q,b^{1-q}): q_1\leq q\leq q_2,\ q_1\neq q_2\}\subset\mathrm{Com}((P,N);\ (a^{t_1},b^{1-t_1}))$. If $h^{-1}(p)\neq\mathrm{Com}((P,N);\ (a^{t_1},b^{1-t_1}))$, we can assume without loss of generality that there exists an $\gamma<q_1$ such that for all t with $\gamma\leq t\leq q_1$ we have $h(a^t,b^{1-t})\in P-\{z\}\subset J-\{z\}$. Hence for each $t\in[\gamma,q_1]$ there exist β_t and γ_t such that $h^{-1}(h(a^t,b^{1-t}))=\{(a^w,b^{1-w}): \beta_t\leq w\leq\gamma_t,\ \beta_t\neq\gamma_t\}$. Moreover for each $s_1,s_2\in S$ with $s_1\neq s_2$ $h^{-1}(s_1)\cap h^{-1}(s_2)=\phi$. Thus $\{h^{-1}(s): s\in\{h(a^t,b^{1-t}): \gamma\leq t\leq q_1\}\}$ is an uncountable collection of disjoint non-degenerate closed intervals contained in the interval $\{(a^t,b^{1-t}): 0\leq t\leq 1\}$. This is impossible. We conclude that $h^{-1}(p)=\mathrm{Com}((P,N);\ (a^{t_1},b^{1-t_1}))$.

2.8 LEMMA. *Let S be rooted with* $P\subset J$ *a positive ideal, and N an ideal of* $(I,\cdot)\times(I,\cdot)$ such that $h^{-1}(P-\{z\})\subset N\subset h^{-1}(P)$ *and* $N\cap h^{-1}(z)$ *closed. Then* $(1,d)\in N^*$ *implies* $(1,c)\in N$ *for all* $0\leq c<d$.

PROOF. Let $(1,d)\in N^*$. One sees immediately that $\{(x,y): 0\leq x<1,\ 0\leq y<d\}\subset N$. Let $p\in P-\{z\}$ and $h^{-1}(p)=\{(a^t,b^{1-t}): t_1\leq t\leq t_2,\ t_1\neq t_2\}$. For $0<c<d$ we have $(1,c)=(1,b^w)$ for some $w\in(0,\infty)$. Hence there exists a q such that $\{(a^{wt},b^{w-wt}): 0<t<q\}\subset N$. If $h(a^{wt},b^{w-wt})=z$, hence $h(1,c)=z$ and thus $(1,c)\in N$. Also if $h(a^{wt'},b^{w-wt'})=p'\in P-\{z\}$ for some $t'\in(0,q)$, then $h(a^{wt},b^{w-wt})=p'$ for all $t\in(0,q)$. Thus $h(1,c)=p'\in P-\{z\}$ and $(1,c)\in N$. For $c=0$, we have $(1,c')\in N$ for $0<c'<d$, and N is an ide:'. Thus $(1,c)=(1,0)\in(1,0)\cdot(1,c')\in N$.

Now we shall investigate the ideal J of S.

2.9 LEMMA. *If* $s\in J$, *then* $sS=sU=sV$ [5].

2.10 DEFINITION. Let $s\in J$, and define $G_s=\{s'$: such that $s'S\cap sS\neq\{z\}$, and $s',s\in J\}\cup\{z\}$. Let $\Sigma=\{G_s: s\in J\}$. Note $s\in J$ implies $s\in G_s$.

2.11 LEMMA. *If* $r,s,w\in S$ *and* $rU\cap sU\neq\{z\}$, *and* $sU\cap wU\neq\{z\}$, *then* $rU\cap wU\neq(z)$.

PROOF. From above we have $ru_1=su_2\neq z$, and $su_3=wu_4\neq z$. Let $u=\min\{u_2,u_3\}$. Then $su\neq z$, and $su\in rU\cap wU$.

2.12 LEMMA. *If $r,s,w\in J$, and $rS\cap sS\neq\{z\}$, and $sS\cap wS\neq\{z\}$, then $rS\cap wS\neq\{z\}$.*

PROOF. From 2.9 we have that $rS=rU$, $sS=sU$, and $wS=wU$. Now apply the previous lemma.

2.13 LEMMA. *If $G_r, G_s\in\Sigma$ and there exists $w\neq z$ such that $w\in G_r\cap G_s$, then $G_r=G_s$.*

PROOF. Let $r'\in G_r$. If $r'=z$, then $r'\in G_s$. If $r'\neq z$, then $r'S\cap rS\neq\{z\}$. Also $wS\cap rS\neq\{z\}$ and $wS\cap sS\neq\{z\}$. Applying 2.12 we see that $r'S\cap wS\neq\{z\}$. Applying 2.12 again we get that $r'S\cap sS\neq\{z\}$. Hence $r'\in G_s$, and $G_r\subset G_s$. By symmetry we get $G_s\subset G_r$. Thus $G_r=G_s$.

2.14 LEMMA. *If $G_s\in\Sigma$, then G_s is an ideal of S.*

PROOF. Suppose $G_s=\{z\}$. Then G_s is an ideal of S. Now suppose $G_s\neq\{z\}$. Let $w\in G_s$, and $r\in S$. If $rw=z$, then by definition $rw=z\in G_s$. If $rw\neq z$, then $w\neq z$, hence $wS\cap sS\neq\{z\}$. Also $rw\in rwS\cap wS\neq\{z\}$. Thus by 2.12 we have $rwS\cap sS\neq\{z\}$. Hence G_s is an ideal of S.

Without much difficulty it can be shown the collection Σ is countable. Also by 2.12 we see that for each $G\in\Sigma-\{\{z\}\}$, G is a positive ideal of S.

2.15 LEMMA. *Let G be a positive ideal of S which is contained in J. Suppose $h(x,y)=z$. If there exist $n\in G-\{z\}$ with $h^{-1}(n)=\{(a^t,b^{1-t}):\ t_1\leq t\leq t_2\}$ and $w>1$, $t'\in[0,1]$, $t''\in[t_1,t_2]$ such that $a^{wt'}=x<a^{t''}$, $b^{w-wt'}=y<b^{1-t''}$, and if $q\geq w$ then for all $(\bar{x},\bar{y})\in(h^{-1}(G)\cap\{(a^{qt},b^{q-qt}):\ 0\leq t\leq 1\})$ we have $h(\bar{x},\bar{y})=z$.*

PROOF. Suppose $h(\bar{x},\bar{y})\neq z$. Then $n, h(\bar{x},\bar{y})\in G-\{z\}$. Hence there exist $s,s'\in S-\{z\}$ such that $ns=h(\bar{x},\bar{y})s'\neq z$. Now $h^{-1}(ns)=h^{-1}(h(\bar{x},\bar{y})\cdot s')\supset\{(a^{rt},b^{r-rt}):\ t_1\leq t\leq t_2,\ t_1\neq t_2\}$, $r\geq q\geq w$. Moreover, there exists $t\in[t_1,t_2]$ such that $0<a^{rt}\leq x$ and $0<b^{r-rt}\leq y$, hence $(\hat{x},\hat{y})\cdot(x,y)=(a^{rt},b^{r-rt})$ for some $(\hat{x},\hat{y})\in(I,\cdot)\times(I,\cdot)$. But $z=h(x,y)\cdot h(\hat{x},\hat{y})=h(a^{rt},b^{r-rt})=ns$. This is a contradiction. Hence $h(\bar{x},\bar{y})=z$.

2.16 LEMMA. *If $G\in\Sigma$ and $G\neq\{z\}$ and $n\in G-\{z\}$, then there exists $u\in U-\{z\}$ such that $nu=z$, and there exists $v\in V-\{z\}$ such that $nv=z$. Moreover u and v can be chosen to be maximum with respect to this property.*

PROOF. Let $n \in G-\{z\}$, and $h^{-1}(n)=\{(a^t, b^{1-t}): t_1 \leq t \leq t_2, t_1 \neq t_2\}$. Since S has zero divisors there exists $(x, y) \in (I, \cdot)\times(I, \cdot)-(\{0\}\times I)\cup(I\times\{0\})$ such that $0<x<a^t$, $0<y<b^{1-t}$ for some $t \in [t_1, t_2]$ and $h(x, y)=z$. Now there exist $x', y' \in (0, 1)$ such that $(x', 1)h^{-1}(n)\cup(1, y')h^{-1}(n)\subset\{(a^{wt}, b^{w-wt}): 0\leq t\leq 1\}$ and $(x, y)\in\{(a^{wt}, b^{w-wt}): 0\leq t\leq 1\}$, $w>1$. Now for $(\bar{x}, \bar{y}) \in (x', 1)\ h^{-1}(n)\cup(1, y')h^{-1}(n)$, $h(\bar{x}, \bar{y})=z$. Select $\hat{x}=\text{lub}\ \{\tilde{x}: (\bar{x}, \bar{y}) \in (\tilde{x}, 1)h^{-1}(n)$ implies $h(\bar{x}, \bar{y})=z\}$ and let $\hat{y}=\text{lub}\{\tilde{y}: (\bar{x}, \bar{y}) \in (1, \tilde{y})h^{-1}(n)$ implies $h(\bar{x}, \bar{y})=z\}$. For $(\bar{x}, \bar{y}) \in (\hat{x}, 1)h^{-1}(n)$ we have $h(\bar{x}, \bar{y})=z$. Also for $(\bar{x}, \bar{y}) \in (1, \hat{y})h^{-1}(n)$, $h(\bar{x}, \bar{y})=z$. Also setting $u=h(\hat{x}, 1)$ and $v=h(1, \hat{y})$; $u \in U-\{z\}$, $v \in V-\{z\}$, $nu=z$, $nv=z$ and u and v are maximum with respect to this property.

Let $G \in \Sigma$ and $G\neq\{z\}$. Select $n \in G-\{z\}$; there exist $c, d \in (0, 1)$ such that $h^{-1}(n)=\{(c^t, d^{1-t}): t_1\leq t\leq t_2, t_1\neq t_2\}$. By 2.16 there exists $u \in U-\{z\}$ such that $nu=z$, but $u<u'\leq i$, $nu'\neq z$. Let $h(x, 1)=u$. From lemma 1 [5], $(x, 1)\ \{(c^t, d^{1-t}): t_1\leq t\leq t_2\}\subset\ \{(c^{wt}, d^{w-wt}): 0\leq t\leq 1\}$. Let $a=c^w$, $b=d^w$. It can be shown without any difficulty that the a and b obtained above are independent of the choice of n.

For each $G_i \in \Sigma$ and $G_i\neq\{z\}$, find $a_i, b_i \in (0, 1)$ in the manner a and b were chosen above. Set $M_i=h^{-1}(G_i)-\{(x, y): h(x, y)=z$ and $a_i^t<x\leq 1$, $b_i^{1-t}<y\leq 1$ for some $t \in [0, 1]\}$.

2.17 LEMMA. *M_i is an ideal of $(I, \cdot)\times(I, \cdot)$ and $M_i\cap h^{-1}(z)$ is closed.*

PROOF. Let $(x, y) \in M_i$ and $(x', y') \in (I, \cdot)\times(I, \cdot)$. If $h(x, y)=z$ then $0\leq x\leq a_i^t$ and $0\leq y\leq b_i^{1-t}$ for some $t \in [0, 1]$. Hence $h(xx', yy')=z$, and $0\leq xx'\leq a_i^t$, $0\leq yy'\leq b_i^{1-t}$, that is, $(xx', yy') \in M_i$. Suppose $h(x, y)\neq z$. If $h(xx', yy')\neq z$, then $(xx', yy') \in M_i$. If $h(xx', yy')=z$, we must show $0\leq xx'\leq a_i^t$, $0\leq yy'\leq b_i^{1-t}$ for some $t \in [0, 1]$. Suppose $0\leq a_i^t<xx'$ and $0\leq b_i^{1-t}<yy'$ for some $t \in [0, 1]$. We have $(xx', yy') \in \{(a^{qt}, b^{q-qt}): 0\leq t\leq 1\}$ and $0<q<1$. From the construction of M_i, there exists $(\hat{x}, \hat{y}) \in \{(a^{rt}, b^{r-rt}): 0\leq t\leq 1\}$, $0<q<r<1$, $h(\hat{x}, \hat{y}) \in G-\{z\}$. But by 2.15, $h(\hat{x}, \hat{y})=z$. This is a contradiction. Thus $(xx', yy')\in M_i$ and M_i is an ideal.

Let $(x, y) \notin M_i\cap h^{-1}(z)$. Then either $h(x, y)\neq z$ or $h(x, y)=z$ and there exists $t \in [0, 1]$ such that $a_i^t<x\leq 1$ and $b_i^{1-t}<y\leq 1$. If $h(x, y)\neq z$, then there exists a neighborhood Q of (x, y) in $(I, \cdot)\times(I, \cdot)$ such that $z \notin h(Q)$. If $h(x, y)=z$, then we

need only show that there exists a neighborhood Q of (x,y) such that $Q\cap M_i=\phi$. Suppose to the contrary that $Q\cap M_i\neq\phi$ for every neighborhood Q of (x,y). Then $[0,x)\times[0,y)\subset M_i$. Let $(\bar{x},\bar{y})\in(a_i^t,x)\times(b_i^{1-t},y)$. Then $h(\bar{x},\bar{y})=z$, $(\bar{x},\bar{y})\in M_i$ and $(\bar{x},\bar{y})\in\{(a^{qt},b^{q-qt}): 0\leq t\leq 1\}$ and $0<q<1$. This is impossible. Thus $M_i\cap h^{-1}(z)$ is closed in $(I,\cdot)\times(I,\cdot)$.

2.18 REMARK. For $G_i\in\Sigma$ and $G_i\neq\{z\}$, we note that the definition of M_i gives us $h^{-1}(G_i-\{z\})\subset M_i\subset h^{-1}(G_i)$, and $M_i\cap h^{-1}(z)$ is closed in $(I,\cdot)\times(I,\cdot)$. Thus 2.7 and 2.8 apply to M_i.

2.19 LEMMA. *Let* $M_0=h^{-1}(z)$. *If* $i\neq j$, *then* $M_i^*\cap M_j^*\subset M_0$.

PROOF. Let $(x,y)\in M_i^*\cap M_j^*$, $i\neq j$. If $h(x,y)\neq z$, then there exist $x',y'\in(0,1)$ such that $(xx',yy')\in M_i\cap M_j$ and $h(xx',yy')\neq z$. But this implies $h(xx',yy')\in G_i\cap G_j-\{z\}$. This is impossible. Thus $h(x,y)=z$ and $M_i^*\cap M_j^*\subset M_0$ for $i\neq j$.

The ideals M_i^*, M_0 for $i=1, 2, 3, \cdots$, are closed ideals of $(I,\cdot)\times(I,\cdot)$. Also $M_0=M_0^*$.

Now consider the collection $\mathscr{M}(S)$ where $\mathscr{M}(S)=\{M_i^*, (a_j,b_j): i=0, 1, 2, 3, \cdots$, and $j=1, 2, 3, \cdots\}$.

From the construction of $\mathscr{M}(S)$ and 2.19 we see that $\mathscr{M}(S)$ satisfies the hypothesis of 1.2 and thus we have the following result:

2.20 LEMMA. $[(I,\cdot)\times(I,\cdot)]/R(\mathscr{M}(S))$ *is a semigroup.*

2.21 DEFINITION. We will say that S is an *A-semigroup* if S satisfies property (β) and the boundary of S is the union of two usual unit intervals.

2.22 THEOREM. *If S is an A-semigroup, then* $\mathscr{M}(S)$ *is of Class A and hence* $[(I,\cdot)\times(I,\cdot)]/R(\mathscr{M}(S))$ *is an A-semigroup.*

PROOF. In view of what has already been shown, we need only show that $M_i^*\cap(\{0\}\times I\cup I\times\{0\})\subset\{(0,1),(1,0)\}$ for $i>0$. Let $(x,1)\in M_i^*$ and $x>0$. Then by 2.8 we see that $(x',1)\in M_i$ for $0<x'<x$. Now the restriction of h from $I\times\{1\}$ onto U is an iseomorphism, hence $h(x',1)\in G_i-\{z\}$. Fix $x'\in(0,x)$ and let $(\bar{x},\bar{y})\in(I,\cdot)\times(I,\cdot)-(I\times\{1\}\cup\{1\}\times I)$ with $h(\bar{x},\bar{y})=h(x',1)$. Since S has zero divisors there exists $(c,d)\in(I,\cdot)\times(I,\cdot)-(\{0\}\times I)\cup(I\times\{0\})$ such that $h(c,d)=z$. Now there exists $n\in\{1, 2, 3, \cdots\}$ such that $0<\bar{x}^n<c$ and $0<\bar{y}^n<d$. Hence $h(x'^n, 1)=h(\bar{x}^n,\bar{y}^n)=h(p,q)\cdot h(c,d)=z$ for some $(p,q)\in(I,\cdot)\times(I,\cdot)$. This is a contradiction.

Thus if $(x,1)\in M_i^*$, $x=0$. A similar argument can be used to show that if $(1,y)\in M_i^*$, then $y=0$. We conclude that $M_i^*\cap(\{0\}\times I\cup I\times\{0\})\subset\{(0,1),(1,0)\}$.

2.23 DEFINITION. A semigroup S is said to be a *B-semigroup* if S satisfies property (β) and the boundary of S is the union of a usual unit interval and a nil thread.

2.24 THEOREM. *If S is a B-semigroup then $\mathscr{M}(S)$ is of Class B and hence $[(I,\cdot)\times(I,\cdot)]/R(\mathscr{M}(S))$ is a B-semigroup.*

PROOF. We need only show that $M_i^*\cap(\{1\}\times I)\subset\{(1,0)\}$ and $M_0\cap(\{1\}\times I\cup\{1\}\times I)=[0,1/2]\times\{1\}\cup\{(1,0)\}$. The latter holds since $M_0=h^{-1}(z)$. Now suppose $(1,y)\in M_i^*\cap(\{1\}\times I)$ and $y>0$. Then by 2.8 $(1,y')\in M_i$ for $0<y'<y$. Now $h|\{1\}\times I\to U$ is an iseomorphism, hence $h(1,y')\in G_i-\{z\}$. Fix $y'\in(0,y)$ and let $(\bar{x},\bar{y})\in(I,\cdot)\times(I,\cdot)$ with $h(\bar{x},\bar{y})=h(1,y')$. Since $M_0\supset[0,1/2]\times I$ there exists an $n\in\{1,2,3,\cdots\}$ such that $(\bar{x}^n,\bar{y}^n)\in M_0$. But $h(1,y'^n)=h(\bar{x}^n,\bar{y}^n)=z$. This is a contradiction. Thus $y=0$ and $M_i^*\cap(\{1\}\times I)=\{(1,0)\}$.

2.25 DEFINITION. A *C-semigroup* is a semigroup S satisfying property (β) and the boundary of S is the union of two nil threads.

2.26 THEOREM. *If S is a C-semigroup, then $\mathscr{M}(S)$ is of Class C and hence $[(I,\cdot)\times(I,\cdot)]/R(\mathscr{M}(S))$ is a C-semigroup.*

PROOF. Since $M_0^*=M_0=h^{-1}(z)$ we get $M_0^*\cap(\{1\}\times I\cup I\times\{1\})=\{1\}\times[0,1/2]\cup[0,1/2]\times\{1\}$. Hence we need only show that if $(x,1),(1,y)\in M_i^*$, then $0\leq x\leq 1/2$ or $0\leq y\leq 1/2$. Let $(x,1)$, $(1,y)\in M_i^*$ and $1/2<x\leq 1$ and $1/2<y\leq 1$. Then $(x',1)$, $(1,y')\in M_i$ for $1/2<x'<x\leq 1$ and $1/2<y'<y\leq 1$. Hence $h(x',1)$, $h(1,y')\in G_i-\{z\}$. This implies that there exist $s,s'\in S-\{z\}$ such that $h(x',1)s=h(1,y')s'\neq z$. Hence by 2.9 there exist $\bar{x},\bar{y}\in[0,1]$ such that $h(x',1)h(\bar{x},1)=h(1,y')h(1,\bar{y})\neq z$. Thus $h(x'\bar{x},1)=h(1,y'\bar{y})\neq z$. This contradicts the fact that $U\cap V=\{z,i\}$. We conclude that if $(x,1),(1,y)\in M_i^*$ that $0\leq x\leq 1/2$ or $0\leq y\leq 1/2$.

2.27 REMARK. If S satisfies property (β), then S is an A-semigroup, B-semigroup, or C-semigroup. Hence $\mathscr{M}(S)$ is of Class A, Class B, or Class C, respectively.

2.28 THEOREM. *If S satisfies property (β), then $[(I,\cdot)\times(I,\cdot)]/R(\mathscr{M}(S))$ satisfies property (β). Moreover, $[(I,\cdot)\times(I,\cdot)]/R(\mathscr{M}(S))$ is iseomorphic to S.*

PROOF. Consider the following diagram:

$$\begin{array}{ccc} (I,\cdot)\times(I,\cdot) & \xrightarrow{\quad h \quad} & S \\ \phi \big\downarrow & \nearrow_{h\phi^{-1}} & \\ [(I,\cdot)\times(I,\cdot)]/R(\mathscr{M}(S)) & & \end{array}$$

To show that $h\phi^{-1}$ is an iseomorphism we need only show that $\phi^{-1}\ \phi(x,y)=h^{-1}$ $h(x,y)$ for all $(x,y)\in(I,\cdot)\times(I,\cdot)$. If $(x,y)\in M_0=h^{-1}(z)$, then $\phi^{-1}\phi(x,y)=M_0$ and $h^{-1}(h(x,y))=h^{-1}(z)=M_0$. Also if $\phi^{-1}\phi(x,y)=\{(x,y)\}$, then $h(x,y)\notin J$, thus $h^{-1}(h(x,y))=\{(x,y)\}$. Now suppose $(x,y)\notin M_0$ and $\phi^{-1}(\phi(x,y))$ is not a point. Then $\phi^{-1}(\phi(x,y))=$the component of $(M_i^*\cap\{(a_i^{tw},b_i^{w-wt})$: $0\leq t\leq 1$, for some $w\in(0,1)$ containing $(x,y)=\{(a_i^{wt},b_i^{w-wt})$: $t_1\leq t\leq t_2$, $t_1\neq t_2\}$. Hence $\{(c,d)$: $c<a_i^{wt}$, $d<b_i^{w-wt}$ for some $t\in[t_1,t_2]\}\subset M_j$. Let $w_n\to w$, $0<w<w_n<1$. We have $a_i^{w_nt}<a_i^{wt}$, $b_i^{w_n-w_nt}<b_i^{w-wt}$ for $t\in[t_1,\ t_2]$. This implies $\{(a_i^{w_nt},b_i^{w_n-w_nt})$: $t_1\leq t\leq t_2\}\subset M_i$. From 2.8, $h(a_i^{w_nt_1},b_i^{w_n-w_nt})=h(a_i^{w_nt},b_i^{w_n-w_nt})=h(a_i^{w_nt_2},b_i^{w_n-w_nt})$ for $t_1\leq t\leq t_2$. Hence $\lim h(a_i^{w_nt}$, $b_i^{w_n-w_nt})=h(a_i^{wt},b_i^{w-wt})=h(x,y)$ for $t_1\leq t\leq t_2$. Thus $h^{-1}(h(x,y))=\{(a_i^{wt},b_i^{w-wt})$: $t_1\leq t\leq t_2\}$ and $\phi^{-1}(\phi(x,y))=h^{-1}(h(x,y))$. The induced homomorphism theorem implies that $h\phi^{-1}$ is an iseomorphism.

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BIBLIOGRAPHY

[1] Borrego, J.T., Cohen, H., and DeVun, E.E., *Uniquely Representable Semigroups.* Pac. J. Math., to appear.

[2] Brown, D.R., *Topological semilattices on the two-cell.* Pac. J. Math., 15 (1965), No. 1, 35—46.

[3] Clark, C.E., *Homomorphisms of compact semigroups.* Thesis, Louisiana State University, 1966.

[4] Cohen, H. and Collins, H.S., *Affine semigroups.* Trans. Amer. Math. Soc., 93

(1959), 97—113.

[5] DeVun, E.E., *Special Semigroups on the Two-cell*. Pac. J. Math., 34 (1970), No. 3, 639—645.

[6] Hildebrandt, J.S., *On uniquely divisible semigroups on the two-cell*. Pac. J. Math., 23 (1967), 91—95.

[7] Hoffman, K. and Mostert, P.S., *Elements of Compact Semigroups*. Columbus, Ohio; Charles E. Merrill Books, Inc., 1966.

[8] Whyburn, G.T., *Analytic Topology*. Providence, Rhode Island; American Mathematical Society Colloquium Publications, Vol. XXVIII, 1942.

(1) For futher information concerniog semigroup on N-cells one should see a new paper by Klaus Keimel, "Congruence Relations on Cone Semigroups", Semigroup Forum 3 (1971), 130—142.

(2) This paper contains part of a doctoral dissertation written under the direction of professor Hankell Cohen.