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ON ASCOLI THEOREMS AND THE PRODUCT OF k-SPACES

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Let $F \subset C(X, Y)$ be endowed with the compact open topology. If X is locally compact and regular and if Y is a regular or uniform Hausdorff space, then the Ascoli Theorems [6, pp. 233-236] give us necessary and sufficient conditions in order that F be compact. Kelley [6, pp. 234, 236] observes that the necessary and sufficient conditions must be modified if they are to apply when X is a regular or Hausdorff k-space. In this paper we show that the necessary and sufficient conditions apply without modification when X is a k-space which need not be regular or Hausdorff; in fact, we shall even use a definition of k-space which is more encompassing than that used by Kelley.

1. Local compactness. We can define local compactness of a topological space X in several ways which in general are not equivalent.

DEFINITION 1.1 A space X is *locally compact* iff each $x \in X$ has at least one compact neighborhood.

DEFINITION 1.2 A space X is *locally compact* iff each $x \in X$ has at least one closed compact neighborhood.

DEFINITION 1.3 A space X is *locally compact* iff each neighborhood of each point contains a compact neighborhood of that point.

DEFINITION 1.4 A space X is *locally compact* iff each neighborhood of each point contains a closed compact neighborhood of that point, (thus the space is also regular).

In Hausdorff or regular spaces the four definitions above are equivalent since a locally compact (1.1) Hausdorff space is regular, [6, p. 146]. Clearly

$$\begin{array}{ccccc} & \nearrow & (1.3) & \searrow & \\ (1.4) & & & & (1.1) \\ & \searrow & (1.2) & \nearrow & \end{array}$$

Examples are available which show that none of the other possible implications hold, [5, 7].

2. k-spaces. We can also define k-spaces in two different ways.

DEFINITION 2.1. A topological space X is called a *k-space* if A is closed when $A\cap K$ is closed for every closed compact $K\subset X$.

DEFINITION 2.2. A topological space X is called a *k-space* if A is closed when $A\cap K$ is closed relative to K for every compact $K\subset X$.

Clearly the class of k-spaces (2.2) is larger than the class of k-spaces (2.1). For example, give R the topology generated by the intervals $(-\infty, a)$ and let $A=(-\infty, 0]$. A is not closed yet A meets each compact closed set in a closed set since ϕ is the only compact closed set. Thus we do not have a k-space (2.1). The sets of the form $(-\infty, a]$ are compact so that if a set B meets every compact set in a relatively closed set, then B must be of the form $[b, \infty)$ and hence B must be closed. Thus this space is a k-space (2.2). Note that this space is locally compact (1.3) so that even local compactness (1.3) is not enough to imply the equivalence of definitions (2.1) and (2.2).

LEMMA 2.3. *Let X be a topological space in which the closure of every compact set is compact. Then if $A\subset X$, the following are equivalent.*

a) $A\cap K$ is closed for every closed compact $K\subset X$.

b) $A\cap K$ is closed relative to K for every compact $K\subset X$.

PROOF. b)$\rightarrow$a) trivially. Assume that a) holds and that there is a compact $K^*\subset X$ such that $A\cap K^*$ is not closed relative to K^*. Then there is an $x\in \mathrm{Cl}_{K^*}(A\cap K^*)\setminus(A\cap K^*)$. Note that $x\in K^*$ so that $x\notin A$. But $K=\mathrm{Cl}(K^*)$ is compact and closed so that $A\cap K$ is closed. Each neighborhood of x meets $A\cap K^*$ and hence also meets $A\cap K$ so that $x\in \mathrm{Cl}(A\cap K)=A\cap K$. This contradicts the fact that $x\notin A$. Thus a)$\rightarrow$b).

Thus, we observe that the above definitions of k-space are equivalent for any space X in which the closure of every compact set is compact (e.g. in regular, Hausdorff, or KC spaces [9]).

LEMMA 2.4. *Every closed subset of a k-space* (2.1) *or* (2.2) *is a k-space* (2.1 *resp.* 2.2).

THEOREM 2.5. *A topological space X is a k-space* (2.2) *if each point of X has a neighborhood which is a k-space* (2.2). *A similar result is obtained regarding k-spaces* (2.1) *if we demand that the neighborhoods be closed.*

PROOF. Let C be a subset of X such that $C \cap K$ is K-closed for every compact $K \subset X$. Let $x \in \mathrm{Cl}(C)$ and let W be a k-space neighborhood of x. If $K \subset W$ is compact in W, then K is compact in X so that $C \cap K$ is K-closed. Thus there is a closed set C^* such that $C \cap K = C^* \cap K$. Then $(W \cap C) \cap K = (W \cap C^*) \cap (C \cap K)$ is the intersection of a W-closed set with a K-closed set so that $(W \cap C) \cap K$ is a K-closed set. Since W is a k-space (2.2) it follows that $W \cap C$ is W-closed. But since every neighborhood of x clearly meets $W \cap C$ we have $x \in \mathrm{Cl}(W \cap C)$ so that $x \in W \cap \mathrm{Cl}(W \cap C) = \mathrm{Cl}_W(W \cap C) = W \cap C$ and hence $x \in C$. Thus C is closed and X is a k-space (2.2).

Observing that a compact space is always a k-space we obtain the following corollaries.

COROLLARY 2.6. *If X is a locally compact space, then X is a k-space* (2.2).

COROLLARY 2.7. *If X is a locally compact* (1.2 *or* 1.4) *space then X is a k-space* (2.1).

In fact, Corollary 2.6 is a special case of a result due to D.E. Cohen [2] which says that a space is a k-space (2.2) if and only if it is a quotient space of some locally compact (1.1 or 1.2) space. The analogous characterization of k-spaces (2.1)-hopefully in terms of local compactness (1.3 or 1.4)-remains open.

LEMMA 2.8. *Assume that for any $A \subset X$, each point in $Cl(A)$ is the limit of some sequence in A. Then X is a k-space* (2.2).

PROOF. Let C meet each compact subset of X in a relatively closed set. We must show that C is closed. If $x \in \mathrm{Cl}(C) \setminus C$, let $x = \lim x_n$ for some sequence $x_n \in C$. $K = \{x\} \cup \{x_n : n = 1, 2, \cdots\}$ is compact and $C \cap K = \{x_n : n = 1, 2, \cdots\}$ is not K-closed. Thus $\mathrm{Cl}(C) \setminus C = \phi$ and C is closed.

COROLLARY 2.9. *Every topological space satisfying the first axiom of countability is a k-space* (2.2).

3. The product of k-spaces. It is known that the product of k-spaces need not be a k-space. Cohen [3] has shown that if X is a k-space (2.2) whose compact sets are regular and if Y is locally compact and regular, then $X \times Y$ is a k-space (2.2). Fundamental to Cohen's proof is a lemma of Whitehead [8]. Observing that Whitehead's lemma can be proved under the weakened assumption that "each point in a saturated open set $V \subset Q$ is contained in a saturated open set which is contained in some compact subset of V", we can strengthen Cohen's result to the following.

THEOREM 3.1. *If Y is locally compact* (1.3) *and X is a k-space* (2.2) *whose compact subsets are locally compact* (1.3), *then $X\times Y$ is a k-space* (2.2).

We will now eliminate the condition that the compact subsets of X be locally compact.

LEMMA 3.2. *Let X_1, Y_1 be subsets of topological spaces X, Y respectively and let $C\subset X_1\times Y_1$ be closed with respect to $X_1\times Y_1$. Then if π_A denotes projection into A,*

i) $\pi_X(C)$ is closed with respect to X_1 if Y_1 is compact

ii) $\pi_Y(C)$ is closed with respect to Y_1 if X_1 is compact.

PROOF. See Dugundji [4] p. 228.

THEOREM 3.3. *Let X be a k-space* (2.2) *and Y be locally compact* (1.3). *Then $X\times Y$ is a k-space* (2.2).

PROOF. The proof is patterned after that of Bagley and Yang [1]. Let C be a subset of $X\times Y$ which meets every compact subset K of $X\times Y$ in a K-closed set. Let $(x, y)\in \mathrm{Cl}(C)$, V be a compact neighborhood of y and $U\subset V$ another compact neighborhood of y. Define $T=\pi_Y[C\cap(\{x\}\times V)]$ and $S=\pi_X[C\cap(X\times U)]$. If A is any compact subset of X, then $S\cap A=\pi_X[C\cap(A\times U)]$ is A-closed by the above lemma since A and U are compact and $C\cap(A\times U)$ is therefore closed with respect to $A\times U$. Hence, S is closed since X is a k-space (2.2). If W is any neighborhood of x, then $W\times U$ is a neighborhood of (x, y) and so $C\cap(W\times U)\neq\phi$. Then $S\cap W=\pi_X[C\cap(W\times U)]\neq\phi$ so that $x\in\mathrm{Cl}(S)=S$.

Using the above lemma again we see that T is V-closed. Then, since $x\in S$ there is a $y^*\in U\subset V$ such that $(x, y^*)\in C$. Then $y^*\in U\cap T$: i.e. $U\cap T\neq\phi$ whenever $U\subset V$ is a compact neighborhood of y. Since Y is locally compact (1.3) it follows that $y\in\mathrm{Cl}(T)\cap V=\mathrm{Cl}_V(T)=T$ since T is V-closed. But then $(x, y)\in C$ and the proof is complete.

THEOREM 3.4. *Let X be a k-space* (2.1) *in which each point has compact closure (e.g. a T_1-space) and let Y be locally compact and regular; i.e., locally compact* (1.4). *Then $X\times Y$ is a k-space* (2. 1).

PROOF. In the above proof we consider only compact closed $K\subset X\times Y$, U, V, A are to be compact and closed. As above we find that $x\in S$.

Then $\mathrm{Cl}\{x\}\times V$ is compact and closed so that $C\cap(\mathrm{Cl}\{x\}\times V)$ is closed and hence $C\cap(\{x\}\times V)$ is closed with respect to $\{x\}\times V$. Using lemma 3.2 we see that T is

V-closed and hence closed. The proof continues as in 3.3.

4. **Ascoli theorems.** In this section C will denote the family of all continuous functions from a topological space X to a topological space Y. $\mathscr{C}$ will denote the compact open topology for C. Recall the Ascoli theorems given in Kelley [6, pp 233, 236].

THEOREM 4.1. *If Y is a Hausdorff uniform space, X is locally compact and regular, and $F\subset C$, then F is compact in $(C, \mathscr{C})$ if and only if*

a) F is closed in $(C, \mathscr{C})$,

b) $F(x)$ has compact closure for each $x\in X$,

c) F is equicontinuous.

THEOREM 4.2. *If Y is a Hausdorff regular space, X is locally compact and regular, and $F\subset C$, then F is compact in $(C, \mathscr{C})$ if and only if*

a) F is closed in $(C, \mathscr{C})$,

b) $F(x)$ has compact closure for each $x\in X$.

c) F is evenly continuous.

Kelley [6] observes that the above theorems hold true for Hausdorff or regular k-spaces X provided we modify (c) to read: F is equicontinuous (evenly continuous) on each compact subset of X. Bagley and Yang [1] observe that (4.1) and (4.2) hold true for Hausdorff k-spaces X with no modification whatsoever. In this section, we show that the Ascoli Theorems (4.1) and (4.2) hold true for the larger class of (2.2) k-spaces X with no modification. To do so, we need the following lemmas.

LEMMA 4.3. *Let X and Y be topological spaces such that X is Hausdorff or regular or Y is regular. If $\mathscr{T}\supset\mathscr{C}$ and $(F, \mathscr{T})\times X$ is a k-space (2.2), then $\mathscr{T}$ is jointly continuous.*

PROOF. Let A be a closed subset of Y. We are to show that $e^{-1}(A)$ is closed in $F\times X$ where e denotes the evaluation map from $F\times X$ into Y. To do this it suffices to show that $K\cap e^{-1}(A)$ is K-closed for every compact $K\subset F\times X$. Let $M=K\cap e^{-1}(A)$. We are to show that if $(f, x)\in K$ and $(f, x)\notin M$, then $(f, x)\notin \mathrm{Cl}_K(M)$; i.e., M is K-closed. If $(f, x)\in K\setminus M$, then $(f, x)\notin e^{-1}(A)$. Let $U=Y\setminus A$ and K_X be the projection of K into X. Then U is open, $f(x)\in U$, K_X is compact and $x\in K_X$. If X is Hausdorff or regular then K_X is regular and by continuity of f, there is a closed compact neighborhood N of x in the space K_X such that $f(N)\subset U$. If Y is regular then there is a closed neighborhood $U^*\subset U$ of $f(x)$: $N=$

$K_X \cap f^{-1}(U^*)$ is then a closed compact neighborhood of x in the space K_X. In any case $x \in N$, N is compact, and $f(N) \subset U$. Now $[N, U] = \{g : g \in F \text{ and } g(N) \subset U\} \in \mathscr{C} \subset \mathscr{T}$ so that $[N, U]$ is a $\mathscr{T}$-neighborhood of f and $e([N, U] \times N) \subset U$. It follows that $([N, U] \times N) \cap e^{-1}(A) = \phi$: hence $(f, x) \notin \mathrm{Cl}_{F \times K_x}(M) = \mathrm{Cl}(M) \cap (F \times K_X) \supset \mathrm{Cl}(M) \cap K = \mathrm{Cl}_K(M)$. Thus M is closed in the space K and the proof is complete.

LEMMA 4.4. *Let X and Y be topological spaces such that X is Hausdorff or regular or Y is regular. If X is a k-space* (2. 2) *and $(F, \mathscr{C})$ is locally compact* (1.3), *then $\mathscr{C}$ is jointly continuous.*

PROOF. By Theorem (3.3), $(F, \mathscr{C}) \times X$ is a k-space (2.2). The conclusion then follows from the preceding lemma.

COROLLARY 4.5. *Observe that if Y is Hausdorff or regular, we need only assume local compactness* (1.1) *on $(F, \mathscr{C})$ since the properties of Hausdorff separation or regularity are carried over to $(F, \mathscr{C})$.*

THEOREM 4.6. *Let Y be a Hausdorff uniform space and X a k-space* (2.2). *Then F is compact in $(C, \mathscr{C})$ if and only if*

a) F is closed in $(C, \mathscr{C})$,

b) $F(x)$ has compact closure for each $x \in X$,

c) F is equicontinuous.

PROOF. The proofs of this and the following theorems are the same as those of Kelley [6, Theorems 7.17 and 7.21] since, by the above lemma, $\mathscr{C}$ is jointly continuous.

THEOREM 4.7. *Let Y be a regular Hausdorff space and X a k-space* (2.2). *Then F is compact in $(C, \mathscr{C})$ if and only if*

a) F is closed in $(C, \mathscr{C})$,

b) $F(x)$ has compact closure for each $x \in X$,

c) F is evenly continuous.

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