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ON SOME PROBLEMS LEADING TO GENERALIZED HYPERGEOMETRIC AND LAGUERRE POLYNOMIALS

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The author continuous his studies of generalized hypergeometric polynomials [Proc. Nat. Acad. Sci. India Sect. A 37 (1967), 79—96 ; MR 39♯4454] to obtain some problems on integral and expansion involving Laguerre polynomials in terms of Kampé De Fériet's functions. Some known, new and interesting results for a number of familiar polynomials appearing in the literature of special functions have also illustrated as particular cases.

1. Introduction.

The author [(3), p. 79, (2.1)] has recently defined the generalized hypergeometric polynomial by means of

$$(1.1) \qquad F_n(x)=x^{(m-1)n}\,{}_{p+m}F_q\left[\begin{matrix}\Delta(m,\ -n),\ a_p \\ b_q\end{matrix};\ kx^c\right]$$

where m, n are positive integers; notation $\Delta(m,\ -n)$ stands for m-parameters:

$\frac{-n}{m},\ \frac{-n+1}{m},\ \cdots\cdots,\ \frac{-n+m-1}{m}$; $a_p(b_q)$ or ${}_1a_p({}_1b_q)$ for $p(q)$ parameters $a_1,\ \cdots,\ a_p(b_1,\ \cdots,\ b_q)$.

In the course of an attempt to unify and to extend the investigation of certain sets of polynomials, the polynomial (1.1) has attracted considerable attention in the theory of special functions and applied mathematics. The polynomial is terminating and unrestricted. The terminating nature of (1.1) is governed by the numerator parameters $\Delta(m,\ -n)$. Also, the parameters $a_p(b_q)$ are all independent of x but these can be functions of n so that the polynomial always remains well defined.

In (1.1), setting $m=c=1$ and suitable parameters, it yields

$$(1.2) \quad F_n(x)={}_{p+1}F_q\left[\begin{matrix}-n,\ n+\alpha+\beta+1,\ {}_2a_p \\ 1+\alpha,\ \frac{1}{2},\ {}_3b_q\end{matrix};\ x\right]=\frac{n!}{(1+\alpha)_n}f_n^{(\alpha,\ \beta)}\left(\begin{matrix}{}_2a_p \\ {}_3b_q\end{matrix}:x\right)$$

the generalized Sister Celine polynomial [(3), p. 80, (2.2)] which reduces to the generalized Rice's polynomial $H_n^{(\alpha,\beta)}(\xi, p, x)$ satisfying the differential equation [(4), eqn. (13)] :

$$\{x^2(1-x)D^3+[(p+\alpha+2)x-(4+\xi+\alpha+\beta)x^2]D^2+[p(1+\alpha)+\{n(n+1)-(1+\xi)(\alpha+\beta+2)+n(\alpha+\beta)\}x]\ D+n\xi(n+\alpha+\beta+1)\}\ H_n^{(\alpha,\beta)}(\xi, p, x)=0, \text{ where } D\equiv d/dx.$$

Further setting $\xi=p$ in $H_n^{(\alpha,\beta)}(\xi, p, x)$, it leads to the Jacobi polynomial $P_n^{(\alpha,\beta)}(1-2x)$.

$$(1.3)\qquad F_n(x)={}_1F_1\left[\begin{matrix}-n\\1+\alpha\end{matrix};x\right]=\frac{n!}{(1+\alpha)_n}L_n^{(\alpha)}(x),$$

the generalized Laguerre polynomial.

$$(1.4)\qquad F_n(x)={}_2F_0\left[\begin{matrix}-n,\ n+a-1\\ \cdots\cdots\cdots\end{matrix};-\frac{1}{b}x\right]=\gamma_n(x,\ a,\ b)$$

the generalized Bessel polynomial.

With $m=2$, $c=-2$ and proper choice of parameters in (1.1), we can also obtain the well-known polynomials of Bedient, Hermite, Legendre and Lommel etc.

Kampé de Fériet's Hypergeometric Function.

Hypergeometric function in two variables has been defined by Kampé de Fériet J. [(1)] in the form

$$(1.5)\quad F\left[\begin{array}{c|c|c} m & a_m & \\ l & b_l : b'_l & \\ n & c_n & x,\ y \\ p & d_p\ d'_p & \end{array}\right]=\sum_{r,s=0}^{\infty}\frac{(a_m)_{r+s}(b_l)_r(b'_l)_s}{r!\,s!\,(c_n)_{r+s}(d_p)_r(d'_p)_s}x^r y^s$$

which is absolutely convergent if

(i) $m+l<n+p+1$ for all x and y,

(ii) $m+l=n+p+1$, $m\neq n$, for all values of x and y in the common region of $|x|^{\frac{1}{m-n}}+|y|^{\frac{1}{m-n}}=1$ and $|x|<1$, $|y|<1$, containing origin,

(iii) $m+l=n+p+1$, $m=n$, for $|x|<1$, $|y|<1$ and $(a)_r=a(a+1)(a+2)\cdots\cdots(a+r-1)$, $(a)_0=1$.

In this paper, an integral for the product of two generalized hypergeometric and Laguerre polynomials has evaluated in terms of Kampé de Fériet's function which has further used to establish the expansion formula for the polynomials in series of Kampé de Fériet's functions and Laguerre polynomials. Special cases which provide interesting new and known results for a variety of familiar polynomials in the literature have also exhibited.

2. In this section, we state the known results which will be employed in the present investigation.

(i) Integral [(5), p. 27, (2.1) for $\delta=c=1$] :

$$(2.1)\quad \int_0^\infty e^{-x}x^{\beta-1}L_n^{(\alpha)}(x)\,{}_{p+1}F_q\left(\begin{matrix}-m,\ a_p\\ b_q\end{matrix}:\lambda x\right)dx$$
$$=\frac{\Gamma(\beta)(\alpha-\beta+1)_n}{n!}\,{}_{p+3}F_{q+1}\left(\begin{matrix}-m,\ \beta,\ -\alpha+\beta,\ a_p\\ -\alpha+\beta-n,\ b_q\end{matrix}:\lambda\right),\quad \mathrm{Re}(\beta)>0.$$

(ii) Orthogonality-property [(2), pp. 205—206, (4)&(7)] :

$$(2.2)\quad \int_0^\infty x^\alpha e^{-x}L_n^{(\alpha)}(x)\ L_m^{(\alpha)}(x)dx=\begin{cases}0, & \text{if } m\neq n,\ \mathrm{Re}(\alpha)>-1,\\ \dfrac{\Gamma(1+\alpha+n)}{n!}, & \text{if } m=n,\ \mathrm{Re}(\alpha)>-1,\end{cases}$$

(iii) Relations:

$$(2.3)\quad \frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)}=\frac{(-1)^n}{(\alpha)_n},\quad (\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},\quad (\alpha)_{n-k}=\frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k}$$

for $0\leq k\leq n$.

3. Integral.

Here we evaluate an integral which is required in the development of the expansion formula.

$$(3.1)\quad \int_0^\infty e^{-x}x^{\beta-1}L_n^{(\alpha)}(x)\left\{{}_{p+1}F_q\left(\begin{matrix}-m,\ a_p\\ b_q\end{matrix}:\lambda x\right){}_{p+1}F_q\left(\begin{matrix}-l,\ A_p\\ B_q\end{matrix}:\mu x\right)\right\}dx$$
$$=\frac{\Gamma(\beta)(\alpha-\beta+1)_n}{n!}F\left[\begin{matrix}2\\ p+1\\ 1\\ q\end{matrix}\left|\begin{matrix}\beta,\ -\alpha+\beta\\ -l,\ A_p:-m,\ a_p\\ -\alpha+\beta-n\\ B_q:b_q\end{matrix}\right|\mu,\ \lambda\right],\quad \mathrm{Re}(\beta)>0$$

PROOF. Expressing the polynomial ${}_{p+1}F_q\left(\begin{matrix}-l,\ A_p\\ B_q\end{matrix}:\mu x\right)$ in series on the left hand side, then changing the order of integration and summation, which is justificable due to the absolute convergence of the integral and summation involved in the process, it reduces to

$$(3.2)\quad \sum_{\gamma=0}^{\infty}\frac{(-l)_\gamma(A_p)_\gamma\mu^\gamma}{\gamma!\ (B_q)_\gamma}\int_0^\infty e^{-x}x^{\beta+\gamma-1}L_n^{(\alpha)}(x)\,{}_{p+1}F_q\left(\begin{matrix}-m,\ a_p\\ b_q\end{matrix}:\lambda x\right)dx.$$

On using (2.1) and (2.3), in (3.2), the expression takes the form

$$\frac{\Gamma(\beta)(\alpha-\beta+1)_n}{n!}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\beta)_{r+s}(-\alpha+\beta)_{r+s}}{(-\alpha+\beta-n)_{r+s}}\left\{\frac{(-l)_r(A_p)_r\mu^r}{r!\,(B_q)_r}\right\}\left\{\frac{(-m)_s(a_p)_s\lambda^s}{s!\,(b_q)_s}\right\}$$

which yields the value of the right of (3.1) by virtue of (1.5).

4. Expansion.

Required expansion-formula is

$$(4.1)\quad x^{\beta-1}{}_{p+1}F_q\left(\begin{matrix}-m,\ a_p\\ b_q\end{matrix}:\lambda x\right){}_{p+1}F_q\left(\begin{matrix}-l,\ A_p\\ B_q\end{matrix}:\mu x\right)$$

$$=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+1)}\sum_{r=0}^{\infty}\frac{(-\beta+1)_r}{(\alpha+1)_r}F\left[\begin{matrix}2\\ p+1\\ 1\\ q\end{matrix}\left|\begin{matrix}\beta,\ \alpha+\beta\\ -l,\ A_p;\ -m,\ a_p\\ -r+\beta\\ \beta_q:b_q\end{matrix}\right|\mu,\ \lambda\right]L_r^{(\alpha)}(x)$$

valid for $0<x<\infty$ and $\mathrm{Re}(\alpha+\beta)>0$.

PROOF. Let

$$(4.2)\quad f(x)=x^{\beta-1}{}_{p+1}F_q\left(\begin{matrix}-n,\ a_p\\ b_q\end{matrix}:\lambda x\right){}_{p+1}F_q\left(\begin{matrix}-l,\ A_p\\ B_q\end{matrix}:\mu x\right)=\sum_{r=0}^{\infty}C_rL_r^{(\alpha)}(x),\qquad(0<x<\infty).$$

From (4.2), we can easily obtain C_r in a purely formal manner. With that value of C_r, we have then assumed the series on the right of (4.2) actually converges to $f(x)$, providing $f(x)$ [(2), p.176, §100] is sufficiently well behaved.

Therefore, form (4.2) it follows formally that

$$(4.3)\quad\int_0^{\infty}x^{\alpha+\beta-1}e^{-x}L_n^{(\alpha)}(x){}_{p+1}F_q\left(\begin{matrix}-m,\ a_p\\ b_q\end{matrix}:\lambda x\right){}_{p+1}F_q\left(\begin{matrix}-l,\ A_p\\ B_q\end{matrix}:\mu x\right)dx$$

$$=\sum_{r=0}^{\infty}C_r\int_0^{\infty}x^{\alpha}e^{-x}L_n^{(\alpha)}(x)\,L_r^{(\alpha)}(x)dx,\ \mathrm{Re}(\alpha+\beta)>0.$$

All the integrals on the right in (4.3) vanish except for the single term for which $r=n$. Hence, using (2.2) on the right and (3.1) on the left of (4.3), we have

$$(4.4)\quad C_n=\frac{(-\beta+1)_n\Gamma(\alpha+\beta)}{\Gamma(\alpha+n+1)}F\left[\begin{matrix}2\\ p+2\\ 1\\ q\end{matrix}\left|\begin{matrix}\alpha+\beta,\ \beta\\ -l,\ A_p;\ -m,\ a_p\\ -n+\beta\\ B_q:b_q\end{matrix}\right|\mu,\ \lambda\right],\ \mathrm{Re}(\alpha+\beta)>0$$

which leads to (4.1) with the help of (4.2).

5. Applications.

(a) Particular cases of (3.1) and (4.1):

(i) On using (1.2), we obtain the results for the generalized Sister Celine polynomials

$$(5.1)\quad \int_0^\infty e^{-x}x^{\beta-1}L_n^{(\alpha)}(x)f_n^{(\alpha,\beta)}\binom{{}_2{}^a{}_p}{{}_3{}^b{}_q}:\lambda x\Big)f_l^{(\rho,\sigma)}\binom{{}_2{}^A{}_p}{{}_3{}^B{}_q}:\mu x\Big)dx$$

$$=\frac{\Gamma(\beta)(\alpha-\beta+1)_n(1+\gamma)_m(1+\rho)_l}{n!\ m!\ l!}$$

$$\times F\left[\begin{array}{c|c|c} 2 & -\alpha+\beta,\ \beta & \\ p+1 & -l,\ l+\rho+\sigma+1,\ {}_2A_p;\ -m,\ m+\gamma+\delta+1,\ {}_2a_p & \mu\lambda \\ 1 & -\alpha+\beta-n & \\ q & 1+\rho,\ \frac{1}{2},\ {}_3B_q; 1+\gamma,\ \frac{1}{2},\ {}_3b_q & \end{array}\right]\ \mathrm{Re}\ (\beta)>0.$$

$$(5.2)\quad x^{\beta-1}f_m^{(\gamma,\delta)}\binom{{}_2{}^a{}_p}{{}_3{}^b{}_q}:\lambda x\Big)f_l^{(\rho,\sigma)}\binom{{}_2{}^A{}_p}{{}_3{}^B{}_q}:\mu x\Big)$$

$$=\frac{\Gamma(\alpha+\beta)(1+\gamma)_m(1+\rho)_l}{\Gamma(\alpha+1)m!\ l!}\sum_{r=0}^{\infty}\frac{(-\beta+1)_r}{(\alpha+1)_r}$$

$$\times F\left[\begin{array}{c|c|c} 2 & \alpha+\beta,\ \beta & \\ p+1 & -l, l+\rho+\sigma+1, {}_2A_p; -m,\ m+\gamma+\delta+1,\ {}_2a_p & \mu,\ \lambda \\ 1 & -\gamma+\beta & \\ q & 1+\rho,\ \frac{1}{2},\ {}_3B_q;\ 1+\gamma, \frac{1}{2},\ {}_3b_q & \end{array}\right]L_r^{(\alpha)}(x)$$

where $0<x<\infty$ and $\mathrm{Re}(\alpha+\beta)>0$.

On specializing the parameters, these results can be reduce to the generalized Rice's polynomials which further yield the Jacobi polynomials.

Setting $l=0$ in (5.2), the known result given by the author [(5), p. 29, (3.5)] can be obtained.

(ii) Adjusting the parameters in view of (1.3), we have the relation for the Laguerre polynomials:

$$(5.3)\quad \int_0^\infty e^{-x}x^{\beta-1}L_n^{(\alpha)}(x)\,L_m^{(\gamma)}(\lambda x)\,L_l^{(\rho)}(\mu x)dx$$

$$=\frac{\Gamma(\beta)(\alpha-\beta+1)_n(1+\gamma)_m(1+\rho)_l}{n!\ m!\ l!}F\left[\begin{array}{c|c|c} 2 & \beta,\ -\alpha+\beta & \\ 1 & -l;\ -m & \mu,\ \lambda \\ 1 & -\alpha+\beta-n & \\ 1 & 1+\rho;\ 1+\gamma & \end{array}\right],\ \mathrm{Re}(\beta)>0.$$

(5.4) $x^{\beta-1} L_m^{(\gamma)}(\lambda x) L_l^{(\rho)}(\mu x)$

$$=\frac{\Gamma(\alpha+\beta)(1+\gamma)_m(1+\rho)_l}{\Gamma(\alpha+1)m!\ l!}\sum_{r=0}^{\infty}\frac{(-\beta+1)_r}{(\alpha+1)_r}F\left[\begin{array}{c|c|c} 2 & \alpha+\beta,\ \beta & \\ 1 & -l;\ -m & \\ 1 & -\gamma+\beta & \mu,\lambda \\ 1 & 1+\rho;\ 1+\gamma & \end{array}\right] L_r^{(\alpha)}(x)$$

valid for $0<x<\infty$ and $\mathrm{Re}(\alpha+\beta)>0$.

This is a known result due to the author [(5), p. 30, (3.6)] for $l=0$.

(iii) By the help of (1.4), we have the results associated with the Bessel polynomials

(5.5) $\int_0^{\infty} e^{-x} x^{\beta-1} L_n^{(\alpha)}(x) \gamma_m(x,a,b) \gamma_l(x,A,B)\, dx$

$$=\frac{\Gamma(\beta)(\alpha-\beta+1)_n}{n!}F\left[\begin{array}{c|c|c} 2 & \beta,\ -\alpha+\beta & \\ 2 & -l,\ l+A-1:\ -m,\ m+a-1 & -\frac{1}{B},\ -\frac{1}{b} \\ 1 & -\alpha+\beta-n & \\ 0 & \cdots\cdots;\cdots\cdots & \end{array}\right] \quad \mathrm{Re}(\beta)>0.$$

(5.6) $x^{\beta-1}\gamma_m(x,\ a,\ b)\,\gamma_l(x,\ A,\ B)$

$$=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+1)}\sum_{r=0}^{\infty}\frac{(-\beta+1)_r}{(\alpha+1)_r}F\left[\begin{array}{c|c|c} 2 & \alpha+\beta,\ \beta & \\ 2 & -l, l+A-1:\ -m, m+a-1 & -\frac{1}{B}, -\frac{1}{b} \\ 1 & -\gamma+\beta & \\ 0 & \cdots\cdots;\cdots\cdots & \end{array}\right] L_r^{(\alpha)}(x)$$

where $\mathrm{Re}(\alpha+\beta)>0$ and $0<x<\infty$.

(b) Taking $l=m=0$, $\beta=n+1$ in (4.1), and then using (2.3) etc., we obtain the known result [(2), p. 207, (2)]:

$$x^n=\sum_{k=0}^{n}\frac{(-1)^k n!\,(1+\alpha)_n L_k^{(\alpha)}(x)}{(n-k)!\ (1+\alpha)_k}$$

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