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CERTAIN DOUBLE INTEGRALS

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1. Introduction. In 1931, G.N. Watson [4] proved that the function

$$\tilde{w}_{u,v}(x)=\sqrt{x}\int_0^\infty J_u(x/t)J_v(t)dt/t, \tag{1.1}$$

$R(u+1/2)\geq 0$, $R(v+1/2)\geq 0$ is a fourier kernel.

We say that $g(x)$ is the $\tilde{w}_{u,v}(x)$ transform of $f(x)$ if they satisfy the equation

$$g(x)=\int_0^\infty \tilde{w}_{u,v}(xy)f(y)dy \tag{1.2}$$

If $g(x)=f(x)$ then $f(x)$ is said to be self-reciprocal under the kernel $\tilde{w}_{u,v}(x)$ and is denoted by $R_{u,v}$. If $g(x)=-f(x)$ then $f(x)$ is said to be skew-reciprocal and is denoted by $-R_{u,v}$. The kernel $\tilde{w}_{u,v}(x)$ has the following properties:

(i) $\tilde{w}_{u,v}(x)=\tilde{w}_{v,u}(x)$,

(ii) $\tilde{w}_{u,v}(x)=0(x^{u+1/2}, x^{v+1/2})$ for small x,

$=0(x^{-\frac{1}{4}})$ for large x,

(iii) $\tilde{w}_{u,u-1}(x)=J_{2u-1}(2\sqrt{x})$.

In this chapter we shall evaluate double integrals. I have made use of a number of known functions which are self-reciprocal under the $\tilde{w}_{u,v}(x)$ transform and also have utilised certain results obtained by Singh, B [3]. These results are appended in the beginning. The double integrals evaluated are given in the form of a table.

2(a) The following results are proved by Bhatnagar and Singh, B. [1, 3].

(i) $y^{1/4}I_{-1/4}(y/2)K_{-1/4}(y/2)$ is $R_{-1/4,\ -1/4}$.

(ii) $\sqrt{y}\ J_0(y/\sqrt{2})K_0(y/\sqrt{2})$, $R_{1/2,\ 1/2}$.

(iii) $y^{v/2+u/2+1/2}K_{v/2-u/2}(y)$, $R_{u,v}$.

(iv) $I_0(y)-L_0(y)$, $R_{1/2,\ 1/2}$.

(v) $J_0^2(y/2)+Y_0^2(y/2)$, $R_{-1/2,\ -1/2}$.

(vi) $y^{v/2-u/2+1/2}J_{u/2+v/2}(y)$, $R_{u,v}$.

(vii) $y^{-v}I_v(y/2)K_v(y/2)$, $R_{-v-1/2,3v+1/2}$.

(viii) $y^{-v/2}H_{(1/2)(v-1)}(y)$, $R_{1,v}$.

(ix) $y^v[H_{-v}(y)-Y_{-v}(y)]$, $R_{2v-1/2,1/2}$.

(x) $y^{1/2-u}[J_{u-1/2}(y/2)]^2$, $R_{u,3u-1}$.

(xi) $K_v(y)L_{v-1}(y)+L_v(y)K_{v-1}(y)$, $R_{2v-1/2,3/2}$, $R(v)\geq 0$.

(xii) $y^{-v-1/2}[I_{v+1/2}-L_{v+1/2}(y)]$, $R_{3v+3/2,1/2}$.

(xiii) $\sqrt{y}\,[Y_v(y)H_{v-1}(y)-H_v(y)Y_{v-1}(y)$, $R_{o,2v}$.

(xiv) $y^v[Y_{1/2}(y)H_{-1/2}(y)-H_{1/2}(y)Y_{-1/2}(y)$, $R_{v-1/2,3/2-v}$.

(xv) $y^{u/2+v/2-3/2}S_{-u/2-v/2+1,\,u/2-v/2}(y)$, $R_{u-2,v-2}$.

(xvi) $\pi\sqrt{y}\,[H_o(y)-Y_o(y)]-y^{-1/2}$, $-R_{o,o}$.

(xvii) $\pi\sqrt{y}\,[I_o(y)-L_o(y)]-y^{-1/2}$, $-R_{o,o}$.

2(b) Singh, B. [3] have obtained the following results:

(i) $$\int_0^\infty x^{3/2}(x^2+1)^{-3/2}\tilde{w}_{1/2,1}(xy)dx=\sqrt{2}\,\sin(\sqrt{2y})e^{-\sqrt{2y}}$$

(ii) $$\int_0^\infty H_o(x)\tilde{w}_{1/2,1/2}(xy)=J_o(y)$$

(iii) $$\int_0^\infty x^{-1/2}J_1(2\sqrt{ax})\,K_1(2\sqrt{ax})\tilde{w}_{o,o}(xy)dx=(\sqrt{y}/4a)\log\left(\frac{2a+\sqrt{y^2+4a^2}}{y}\right)$$

(iv) $$\int_0^\infty x^{-1/2}K_{2v}(2\sqrt{ax})\tilde{w}_{-v,v}(xy)dx$$

$$=\frac{y^{v+1/2}\,\Gamma(v+1/2)\Gamma(1/2-v)}{4a^{v+1}}\,{}_2F_1[v+1/2,\ 1/2:1:1-y^2/a^2],$$

$$R(y+a)>0,\ -1/2<R(v)<1/2.$$

(v) $$\int_0^\infty x^\lambda J_{v/2}(x/2)Y_{v/2}(x/2)\tilde{w}_{v-\lambda+1/2,v+\lambda-1/2}\,(xy)dx$$

$$=-\frac{y^{v+\lambda}\Gamma(\lambda+v/2+1/2)}{\sqrt{\pi}\,2^v\Gamma(\lambda+v/2+1)\Gamma(v+1)}\,{}_1F_2[\lambda+v/2+1/2:v+1,\lambda+v/2+1:-y^2/4]$$

(vi) $\int_0^\infty x^{3/2}(1+x^2)^{-3/2}\tilde{w}_{u,1}(xy)dx=2\sqrt{y}J_u(\sqrt{2y})K_u(\sqrt{2y}),\quad R(u)>-1.$

(vii) $\int_0^\infty \sqrt{x}J_v(\sqrt{2ax})K_v(\sqrt{2ax})\tilde{w}_{v,v}(xy)dx$

$$=\frac{\Gamma(v/2+1)\Gamma(v/2+3/2)x^{v+1/2}}{\sqrt{\pi}\Gamma(v+1)a^{v+2}}\ {}_2F_1\left[\begin{matrix}v/2+3/2, v/2+1: \\ v+1:\end{matrix}\ -y^2/a^2\right],$$

$$R(v)\geq -1/2.$$

(viii) $\int_0^\infty \sqrt{x}(a^2+x^2)^{-3/2}\tilde{w}_{0,0}(xy)dx=(2/a)\sqrt{y}K_0(\sqrt{2ay})J_0(\sqrt{2ay})$

(ix) $\int_0^\infty x^{3/2}J_1(x/\sqrt{2})K_1(x/\sqrt{2})\tilde{w}_{1,1}(xy)dx=y^{3/2}J_0(y/\sqrt{2})K_0(y/\sqrt{2})$

(x) $\int_0^\infty \sqrt{x}I_{v/2+u/2}(x/2)K_{v/2-u/2}(x/2)\tilde{w}_{u,v}(xy)dx=e^{-y}/\sqrt{y}$

(ix) $\int_0^\infty y^{v-u-1/2}J_{2v}(2\sqrt{y})\tilde{w}_{u,2v-u}(by)dy$

$$=\frac{2^{2(v-u)}\Gamma(2v-u+1/2)b^{2v-u+1/2}}{\Gamma(v+1)\Gamma(u-v+1/2)(1+b^2)^{2v-u+1/2}}\ {}_2F_1\left[\begin{matrix}v-u/2+1/4, v-u/2+3/4: \\ v+1:\end{matrix}\ \frac{4b^2}{(1+b^2)^2}\right]$$

(xii) $\int_0^\infty x^{-v}J_v(x/2)J_{-v}(x/2)\,\tilde{w}_{3v+1/2,-v-1/2}(xy)dx=-y^{-v}J_v(y/2)Y_v(y/2).$

3. THEOREM. $\int_0^\infty\int_0^\infty f(x^2+y^2)J_{u+v}(2y\sqrt{t})K_{v-u}(2x\sqrt{t})\,dx\,dy$

$=\frac{\pi}{8\sqrt{t}}\sec[(v/2-u/2)\pi]\int_0^\infty \frac{f(z)}{\sqrt{z}}\,\tilde{w}_{u,v}(tz)dz,$ *provided that* $f(z)=0(z^{-1/4-\delta})$ *for large* z *and* $f(z)=0(z^{-u-1+\varepsilon})$ *or* $0(z^{-v-1+\varepsilon})$ *for small* z; $\delta>0$, $\varepsilon>0$: $R(v+u)>-1$, $|R(v-u)|<1$.

PROOF. Singh, B. [3] has proved that

$$\int_0^{\pi/2} J_{u+v}(2rt\sin\theta)K_{v-u}(2rt\cos\theta)\,d\theta$$

$$=\frac{\pi}{4}\sec[(v/2-u/2)\pi]\frac{1}{rt}\,\tilde{w}_{u,v}(r^2t^2).$$

Multiplying both sides by $rf(r^2)$ and integrating with respect to r between the limits $(0, \infty)$ we have

$$\int_0^\infty rf(r^2)dr\int_0^{\pi/2} J_{u+v}(2rt\sin\theta)\, K_{v-u}\,(2rt\cos\theta)\, d\theta$$

$$=\frac{(\pi/4)\sec\left(\frac{v-u}{2}\pi\right)}{t}\int_0^\infty f(r^2)\tilde{w}_{u,v}(r^2t^2)dr$$

On putting $x=r\cos\theta,\ y=r\sin\theta$ we have

$$\int_0^\infty\int_0^\infty f(x^2+y^2)J_{u+v}(2ty)K_{v-u}(2tx)\, dx\, dy$$

$$=\frac{\pi}{4}\frac{\sec[(v/2-u/2)\pi]}{t}\int_0^\infty f(r^2)\tilde{w}_{u,v}(r^2t^2)dr$$

or
$$\int_0^\infty\int_0^\infty f(x^2+y^2)J_{u+v}(2y\sqrt{t})\, K_{v-u}(2x\sqrt{t})dx\, dy$$

$$=\frac{\pi}{8}\sec[(v/2-u/2)\pi]\,(1/\sqrt{t})\int_0^\infty z^{-1/2}f(z)\tilde{w}_{u,v}(tz)dz.$$

The integrals are absolutely convergent on account of the above conditions.

4. EXAMPLES. (1) Let $f(z)=e^{-z}$, then we have from the theorem

$$(\pi/8)\sec[(v/2-u/2)\pi]\,(1/\sqrt{t})\int_0^\infty z^{-1/2}e^{-z}\tilde{w}_{u,v}(tz)dz$$

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}J_{u+v}(2y\sqrt{t})K_{v-u}(2x\sqrt{t})dx\, dy$$

$$=\int_0^\infty e^{-x^2}K_{v-u}(2x\sqrt{t})dx\int_0^\infty e^{-y^2}J_{u+v}(2y\sqrt{t})dy$$

or
$$\int_0^\infty e^{-tz}z^{-1/2}\tilde{w}_{u,v}(z)dz=\frac{2}{\pi}\cos[(v/2-u/2)\pi]$$

$$\times\int_0^\infty e^{-xt}x^{-1/2}K_{v-u}(2\sqrt{x})dx\int_0^\infty e^{-yt}y^{-1/2}J_{u+v}(2\sqrt{y})dy$$

$$=\frac{1}{t}I_{u/2+v/2}(1/2t)K_{v/2-u/2}(1/2t),\quad R(u,v)>-1,\quad -1<R(v-u)<1,$$

which is the result due to Singh, B. [3].

(2) Let $f(z)=z^{3/4}I_{-1/4}(z/2)K_{-1/4}(z/2)$.

Hence from the theorem we have

$$\int_0^\infty\int_0^\infty (x^2+y^2)^{3/4} I_{-1/4}\frac{(x^2+y^2)}{2}K_{-1/4}\frac{(x^2+y^2)}{2}J_{-1/2}(2y\sqrt{t})K_0(2x\sqrt{t})dx\,dy$$

$$=\frac{\pi}{8\sqrt{t}}\sec[(v/2-u/2)\pi]\int_0^\infty z^{1/4}I_{-1/4}(z/2)K_{-1/4}(z/2)\tilde{w}_{-1/4,-1/4}(tz)dz$$

$$=\frac{\pi}{8}t^{-1/4}I_{-1/4}(t/2)K_{-1/4}(t/2),$$ due to the result (i) of 2(a).

Similarly we can obtain the following results with the help of the results of 2(a, b).

(3) $$\int_0^\infty\int_0^\infty (x^2+y^2)J_0\left(\frac{x^2+y^2}{\sqrt{2}}\right)K_0\left(\frac{x^2+y^2}{\sqrt{2}}\right)J_0(2y\sqrt{t})\,K_0(2x\sqrt{t})\,dx\,dy$$

$$=\frac{\pi}{8}J_0(t/\sqrt{2})K_0(t/\sqrt{2}).$$

(4) $$\int_0^\infty\int_0^\infty (x^2+y^2)^{u/2+v/2+1}K_{v/2-u/2}(x^2+y^2)J_{u+v}(2y\sqrt{t})K_{v-u}(2x\sqrt{t})dx\,dy$$

$$=\frac{\pi}{8}\sec[(v/2-u/2)\pi]\,t^{v/2+u/2}K_{v/2-u/2}(t),\quad R(u+v)>-1,\quad |R(v-u)|<1$$

and $\frac{v-u}{2}$ is not an integer.

(5) $$\int_0^\infty\int_0^\infty \sqrt{x^2+y^2}\,[I_0(x^2+y^2)-L_0(x^2+y^2)]J_1(2y\sqrt{t})\,K_0(2x\sqrt{t})dx\,dy$$

$$=\frac{\pi}{8}\,t^{-1/2}[I_0(t)-L_0(t)].$$

(6) $$\int_0^\infty\int_0^\infty \sqrt{x^2+y^2}\,\left[J_0^2\left(\frac{x^2+y^2}{2}\right)+Y_0^2\left(\frac{x^2+y^2}{2}\right)\right]J_1(2y\sqrt{t})K_0(2x\sqrt{t})dx\,dy$$

$$=\frac{\pi}{8}\,t^{-1/2}\,[J_0^2(t/2)+Y_0^2(t/2)].$$

(7) $$\int_0^\infty\int_0^\infty (x^2+y^2)^{v/2-u/2+1}J_{\frac{u+v}{2}}(x^2+y^2)J_{u+v}(2y\sqrt{t})K_{v-u}(2x\sqrt{t})dx\,dy$$

$$=\frac{\pi}{8}\sec[(v/2-u/2)\pi]\,t^{v/2-u/2}J_{u/2+v/2}(t),$$

$R(u+v)>-1$; $R(u-v)>1/2$, $R(v-u)>-1$.

(8) $$\int_0^\infty\int_0^\infty (x^2+y^2)^{1/2-v} I_v\left(\frac{x^2+y^2}{2}\right)K_v\left(\frac{x^2+y^2}{2}\right)J_{2v}(2y\sqrt{t})K_{4v+1}(2x\sqrt{t})dx\ dy$$

$$=(\pi/8)\sec[(2v+1/2)\pi]\,t^{-v-1/2}I_v(t/2)K_v(t/2),\ \ 0>R(v)>-1/2.$$

(9) $$\int_0^\infty\int_0^\infty (x^2+y^2)^{1/2-v/2}H_{v/2-1/2}(x^2+y^2)J_{v+1}(2y\sqrt{t})K_{v-1}(2x\sqrt{t})dx\ dy$$

$$=(\pi/8)\sec[\pi(v/2-1/2)]\,t^{-v/2-1/2}H_{\frac{v-1}{2}}(t),\ \ |R(v-1)|<1.$$

(10) $$\int_0^\infty\int_0^\infty (x^2+y^2)^{v+1/2}[H_{-v}(x^2+y^2)-Y_{-v}(x^2+y^2)]J_{2v}(2y\sqrt{t})$$

$$\times K_{1-2v}(2x\sqrt{t})dx\ dy=(\pi/8)\sec[\pi(1/2-v)]\,t^{v-1/2}[H_{-v}(t)-Y_{-v}(t)],$$
$0<R(v)<1.$

(11) $$\int_0^\infty\int_0^\infty (x^2+y^2)^{1-u}\left[J_{u-1/2}\left(\frac{x^2+y^2}{2}\right)\right]^2 J_{4u-1}(2y\sqrt{t})K_{2u-1}(2x\sqrt{t})dx\ dy$$

$$=(\pi/8)\sec[\pi(u-1/2)]\,t^{-u}[J_{u-1/2}(t/2)]^2,\ \ 1>R(u)>0.$$

(12) $$\int_0^\infty\int_0^\infty \sqrt{x^2+y^2}\ \ [K_v(x^2+y^2)L_{v-1}(x^2+y^2)+L_v(x^2+y^2)K_{v-1}(x^2+y^2)]$$

$$\times J_{2v+1}(2y\sqrt{t})K_{2(1-v)}(2x\sqrt{t})dx\ dy$$

$$=(\pi/8)\sec[\pi(1-v)]\,t^{-1/2}[K_v(t)L_{v-1}(t)+L_v(t)K_{v-1}(t)],\ \ 1/2<R(v)<3/2.$$

(13) $$\int_0^\infty\int_0^\infty (x^2+y^2)^{-v}[I_{v+1/2}(x^2+y^2)-L_{v+1/2}(x^2+y^2)]J_{3v+2}(2y\sqrt{t})$$

$$\times K_{3v+1}(2x\sqrt{t})dx\ dy$$

$$=(\pi/8)\sec[\pi(3v/2+1/2)]\,t^{-v-1}[I_{v+1/2}(t)-L_{v+1/2}(t)],\ -2/3<R(v)<0.$$

(14) $$\int_0^\infty\int_0^\infty (x^2+y^2)\ [Y_v(x^2+y^2)H_{v-1}(x^2+y^2)-H_v(x^2+y^2)Y_{v-1}(x^2+y^2)]$$

$$\times J_{2v}(2y\sqrt{t})K_{2v}(2x\sqrt{t})dx\ dy$$

$$=(\pi/8)\sec(v\pi)\,[Y_v(t)H_{v-1}(t)-H_v(t)Y_{v-1}(t)].$$

(15) $$\int_0^\infty\int_0^\infty (x^2+y^2)^{v+1/2}[Y_{1/2}(x^2+y^2)H_{-1/2}(x^2+y^2)-H_{1/2}(x^2+y^2)$$

$$\times Y_{-1/2}(x^2+y^2)]J_1(2y\sqrt{t})K_{2-2v}(2x\sqrt{t})dx\ dy$$

$$=(\pi/8)\sec[(1-v)\pi]\,t^{v-1/2}[Y_{1/2}(t)H_{-1/2}(t)-H_{1/2}(t)Y_{-1/2}(t)],$$

$1/2<R(v)<3/4.$

(16) $$\int_0^\infty\int_0^\infty (x^2+y^2)^{u/2+v/2-1} S_{-u/2-v/2+1,\,u/2-v/2}(x^2+y^2)$$

$$\times J_{u+v-4}(2y\sqrt{t})K_{v-u}(2x\sqrt{t})dx\ dy$$

$$=(\pi/8)\sec[\pi(v/2-u/2)]\,t^{u/2+v/2-2} S_{-u/2-v/2+1,\,u/2-v/2}(t),$$

$|R(v-u)|<1$, $R(u+v)>-3$, also u and v are note even negative integers.

(17) $$\int_0^\infty\int_0^\infty (x^2+y^2)[H_0(x^2+y^2)-Y_0(x^2+y^2)]J_0(2y\sqrt{t})K_0(2x\sqrt{t})dx\ dy$$

$$=\frac{1}{4t}+\frac{\pi}{8}[Y_0(t)-H_0(t)].$$

(18) $$\int_0^\infty\int_0^\infty (x^2+y^2)[I_0(x^2+y^2)-L_0(x^2+y^2)]J_0(2y\sqrt{t})K_0(2x\sqrt{t})dx\ dy$$

$$=\frac{\pi}{8}[L_0(t)-I_0(t)]+1/4t.$$

(19) $$\int_0^\infty\int_0^\infty (x^2+y^2)^2[(x^2+y^2)^2+1]^{-3/2}J_{3/2}(2y\sqrt{t})K_{1/2}(2x\sqrt{t})dx\ dy$$

$$=\frac{\pi}{4}t^{-1/2}\sin(\sqrt{2t})e^{-\sqrt{2t}}.$$

(20) $$\int_0^\infty\int_0^\infty \sqrt{x^2+y^2}\ \sin[\sqrt{2(x^2+y^2)}]\ e^{-\sqrt{2(x^2+y^2)}}J_{3/2}(2y\sqrt{t})K_{1/2}(2x\sqrt{t})dx\ dy$$

$=\frac{\pi}{8}t(t^2+1)^{-3/2}$; which we get due to the fact that $\tilde{w}_{u,v}(x)$ is a Fourier kernel.

(21) $$\int_0^\infty\int_0^\infty \sqrt{x^2+y^2}\ [H_0(x^2+y^2)J_1(2y\sqrt{t})K_0(2x\sqrt{t})]\ dx\ dy=\frac{\pi}{8\sqrt{t}}J_0(t).$$

(22) $$\int_0^\infty\int_0^\infty \sqrt{x^2+y^2}\ J_0(x^2+y^2)J_1(2y\sqrt{t})K_0(2x\sqrt{t})dx\ dy=\frac{\pi}{8\sqrt{t}}H_0(t).$$

(23) $$\int_0^\infty\int_0^\infty J_1[2\sqrt{a(x^2+y^2)}]K_1[2\sqrt{a(x^2+y^2)}]J_0(2y\sqrt{t})K_0(2x\sqrt{t})dx\ dy$$

$$=\frac{\pi}{32a}\log\left[\frac{2a+\sqrt{t^2+4a}}{t}\right].$$

(24) $$\int_0^\infty\int_0^\infty K_{2v}[2\sqrt{a(x^2+y^2)}]J_0(2y\sqrt{t})K_{2v}(2x\sqrt{t})dx\,dy$$

$$=\frac{\pi\sec(v\pi)\Gamma(v+1/2)\Gamma(1/2-v)t^v}{32a^{v+1}}\,{}_2F_1[v+1/2,\ 1/2;1;1-t^2/a^2],$$

$$R(t+a)>0,\quad -1/2<R(v)<1/2.$$

(25) $$\int_0^\infty\int_0^\infty (x^2+y^2)^{\lambda+1/2}J_{v/2}\left(\frac{x^2+y^2}{2}\right)Y_{v/2}\left(\frac{x^2+y^2}{2}\right)J_{2v}(2y\sqrt{t})$$

$$\times K_{2\lambda-1}(2x\sqrt{t})dx\,dy$$

$$=-\frac{\sqrt{\pi}\Gamma(\lambda+v/2+1/2)\sec[(\lambda-1/2)\pi]\,t^{v+\lambda-1/2}}{2^{v+3}\Gamma(\lambda+v/2+1)\Gamma(v+1)}\,{}_1F_2\left[\begin{matrix}\lambda+v/2+1/2\ ; \\ v+1,\ \lambda+v/2+1\ ;\end{matrix}\ -t^2/4\right],$$

$3/4>R(\lambda)>0$, $R(v)>-1/2$ and λ is not half of an integer.

(26) $$\int_0^\infty\int_0^\infty (x^2+y^2)^2[1+(x^2+y^2)^2]^{-3/2}J_{u+1}(2y\sqrt{t})\,K_{1-u}(2x\sqrt{t})dx\,dy$$

$$=(\pi/4)\sec[\pi(1/2-v)]\ J_u(\sqrt{2t})\,K_u(\sqrt{2t}).$$

(27) $$\int_0^\infty\int_0^\infty (x^2+y^2)J_u[\sqrt{2(x^2+y^2)}]\,K_u[\sqrt{2(x^2+y^2)}]J_{u+1}(2y\sqrt{t})$$

$$\times K_{1-u}(2x\sqrt{t})dx\,dy$$

$$=(\pi/16)\sec[\pi(1/2-u/2)]\,t(1+t^2)^{-3/2}.$$

(28) $$\int_0^\infty\int_0^\infty (x^2+y^2)J_v[\sqrt{2a(x^2+y^2)}]\,K_{2v}[\sqrt{2a(x^2+y^2)}]J_{2v}(2y\sqrt{t})$$

$$\times K_0(2x\sqrt{t})dx\,dy$$

$$=\frac{\sqrt{\pi}\Gamma(v/2+1)\Gamma(v/2+3/2)t^v}{8\Gamma(v+1)a^{v+2}}\,{}_2F_1\left[\begin{matrix}v/2+3/2,\ v/2+1\ ; \\ v+1\ ;\end{matrix}\ -t^2/a^2\right],\quad R(v)>-1/2.$$

(29) $$\int_0^\infty\int_0^\infty (x^2+y^2)\,[a^2+(x^2+y^2)^2]^{-3/2}J_0(2y\sqrt{t})\,K_0(2x\sqrt{t})dx\,dy$$

$$=\frac{\pi}{4a}K_0(\sqrt{2at})\,J_0(\sqrt{2at}).$$

(30) $$\int_0^\infty\int_0^\infty (x^2+y^2)J_0[\sqrt{2a(x^2+y^2)}]K_0[\sqrt{2a(x^2+y^2)}]J_0(2y\sqrt{t})K_0(2x\sqrt{t})dx\,dy$$

$$=\frac{\pi a}{16}(t^2+a^2)^{-3/2}$$

(31) $$\int_0^\infty\int_0^\infty (x^2+y^2)^2 J_1\left(\frac{x^2+y^2}{\sqrt{2}}\right)K_1\left(\frac{x^2+y^2}{\sqrt{2}}\right)J_2(2y\sqrt{t})K_0(2x\sqrt{t})\,dx\,dy$$

$$=\frac{\pi}{8}\,t\,J_0(t/\sqrt{2})K_0(t/\sqrt{2}).$$

(32) $$\int_0^\infty\int_0^\infty (x^2+y^2)^2 J_0\left(\frac{x^2+y^2}{\sqrt{2}}\right)K_0\left(\frac{x^2+y^2}{\sqrt{2}}\right)J_2(2y\sqrt{t})K_0(2x\sqrt{t})dx\,dy$$

$$=\frac{\pi}{8}\,t\,J_1(t/\sqrt{2})K_1(t/\sqrt{2}).$$

(33) $$\int_0^\infty\int_0^\infty (x^2+y^2)I_{v/2+u/2}\left(\frac{x^2+y^2}{2}\right)K_{v/2-u/2}\left(\frac{x^2+y^2}{2}\right)J_{u+v}(2y\sqrt{t})$$

$$\times K_{v-u}(2x\sqrt{t})dx\,dy$$

$$=(\pi/8t)\sec[\pi(v/2-u/2)]\,e^{-t}.$$

On putting $u=v$ we have

$$\int_0^\infty\int_0^\infty (x^2+y^2)I_v\left(\frac{x^2+y^2}{2}\right)K_0\left(\frac{x^2+y^2}{2}\right)J_{2v}(2y\sqrt{t})K_0(2x\sqrt{t})dx\,dy$$

$$=\frac{\pi}{8t}e^{-t},\quad R(v)>-1/2.$$

(34) $$\int_0^\infty\int_0^\infty (x^2+y^2)^{2v-u+1}[1+(x^2+y^2)^2]^{u-2v-1/2}\;{}_2F_1\left[\begin{matrix}v-u/2+1/4,\ v-u/2+3/4\,;\\ v+1\,;\end{matrix}\right.$$

$$\left.\frac{4(x^2+y^2)^2}{[1+(x^2+y^2)^2]^2}\right]J_{2v}(2y\sqrt{t})K_{2(v-u)}(2x\sqrt{t})dx\,dy$$

$$=\frac{\pi\sec[\pi(v-u)]\,\Gamma(v+1)\Gamma(u-v+1/2)t^{v-u-1}J_{2v}(2\sqrt{t})}{2^{2v-2u+3}\Gamma(2v-u+1/2)}.$$

$$R(2v-u)>-1/4,\quad 1/2>R(u-v)>-1/2.$$

(35) $$\int_0^\infty\int_0^\infty (x^2+y^2)^v J_{u-v+1}[\sqrt{2(x^2+y^2)}]K_{u-v+1}[\sqrt{2(x^2+y^2)}]$$

$$\times J_{u+v}(2y\sqrt{t})K_{v-u}(2x\sqrt{t})dx\,dy$$

$$= \frac{2^{2v-5}\sqrt{\pi}\ \sec\,[\pi(v/2-u/2)]\ \ \Gamma(v/2+u/2+1/2)\ \ \Gamma(v+1/2)t^{-v-1}}{\Gamma(-v/2+u/2+3/2)}$$

$$\times {}_2F_1\left[\begin{matrix} v/2+u/2+1/2,\ v+1/2\ ; \\ u/2-v/2+3/2\ ; \end{matrix}\ -1/t^2\right],\ -1<R(v-u)<1,\ 0<R(u)<2.$$

(36) $$\int_0^\infty\int_0^\infty (x^2+y^2)^{-v+1/2} J_v\left(\frac{x^2+y^2}{2}\right) J_{-v}\left(\frac{x^2+y^2}{2}\right) J_{2v}(2y\sqrt{t})$$

$$\times K_{4v+1}(2x\sqrt{t})\ dx\ dy$$

$$= -(\pi/8)\sec\,[\pi(2v+1/2)]\,t^{-v-1/2} J_v(t/2) Y_v(t/2),\ \ 0>R(v)>-1/2.$$

(37) $$\int_0^\infty\int_0^\infty (x^2+y^2)^{-v+1/2} J_v\left(\frac{x^2+y^2}{2}\right) Y_v\left(\frac{x^2+y^2}{2}\right) J_{2v}(2y\sqrt{t})$$

$$\times K_{4v+1}\,(2x\sqrt{t})dx\ dy$$

$$= -(\pi/8)\ \sec\,[\pi(2v+1/2)]\ \ t^{-v-1/2} J_v(t/2)\,J_{-v}(t/2),\ \ 0>R(v)>-1/2.$$

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