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INTEGRALS INVOLVING LAGUERRE, JACOBI AND HERMITE POLYNOMIALS

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Summary.

The purpose of the present paper is to evaluate certain integrals involving Laguerre, Jacobi and Hermite polynomials. These integrals are very useful in case of expansion of any polynomial in a series of Orthogonal polynomials [1, Theo. 56].

Frequent use will be made of the notations given in Rainville [1].

We shall prove the following:

(1) $$\int_0^\infty e^{-x}x^{\alpha+\beta}L_K^{(\alpha)}(x)dx=\frac{(-1)^K}{K!}\frac{\Gamma(1+\alpha+\beta)\Gamma(1+\beta)}{\Gamma(1+\beta-K)},$$

where $\mathrm{Re}(\alpha+\beta)>-1$ and $0\leq K\leq\beta$:

(2) $$\int_{-1}^1(1-x)^\alpha(1+x)^{\beta+Y}P_n^{(\alpha,\beta)}(x)dx=\frac{2^{\alpha+\beta+Y}\Gamma(n+\alpha+1)\Gamma(\beta+Y+1)}{n!\Gamma(\alpha+\beta+Y+n+2)}$$

$$\times\frac{\Gamma(Y+1)}{\Gamma(Y-n+1)},$$ where $\mathrm{Re}(\alpha)>-1$ and $\mathrm{Re}(\beta+Y)>-1$:

(3) $$\int_{-\infty}^\infty e^{-x^2}x^{2l}H_{2n}(x)dx=\frac{2^{2n}\Gamma(l+1)\Gamma(l+1/2)}{\Gamma(l-n+1)}$$

where $0\leq n\leq l$:

(4) $$\int_{-\infty}^\infty e^{-x^2}x^{2l+1}H_{2n+1}(x)dx=\frac{2^{2n+1}\Gamma(l+1)\Gamma(l+3/2)}{\Gamma(l-n+1)},$$

where $0\leq n\leq l$.

PROOF of (1).

Consider the integral

$$A=\int_0^\infty e^{-x}x^{\alpha+\beta}L_K^{(\alpha)}(x)dx. \tag{1.1}$$

From [1, p. 201(3)] and (1.1), we have

$$A=\sum_{r=0}^K\frac{(-1)^r(1+\alpha)_K}{r!(K-r)!(1+\alpha)_r}\int_0^\infty e^{-x}x^{\alpha+\beta+r}dx,$$

If $\mathrm{Re}(\alpha+\beta)>-1$, then

$$A=\sum_{r=0}^{K}\frac{(-1)^r(1+\alpha)_K}{r!(K-r)!(1+\alpha)_r}\Gamma(\alpha+\beta+r+1),$$

or, $$A=\frac{(1+\alpha)_K\Gamma(\alpha+\beta+1)}{K!}\sum_{r=0}^{K}\frac{(-K)_r(\alpha+\beta+1)_r}{r!(1+\alpha)_r}.$$

If $0\leq K\leq\beta$, then

$$A=\frac{(-1)^K\Gamma(\alpha+\beta+1)\Gamma(\beta+1)}{K!\Gamma(\beta+K+1)}.$$

Hence the result follows.

PROOF of (2).

Consider the integral

$$B=\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta+Y}P_n^{(\alpha,\beta)}(x)dx. \tag{2. 1}$$

From [1, p. 255(4)] and (2. 1), we have

$$B=\sum_{k=0}^{n}\frac{(-1)^K(1+\alpha)_n(1+\alpha+\beta)_{n+K}}{K!(n-K)!(1+\alpha)_K(1+\alpha+\beta)_n 2^K}\int_{-1}^{1}(1-x)^{\alpha+K}(1+x)^{\beta+Y}dx.$$

If $\mathrm{Re}(\alpha)>-1$ and $(\beta+Y)>-1$, then

$$B=2^{\alpha+\beta+Y+1}\frac{\Gamma(1+\alpha+n)\Gamma(\beta+Y+1)}{n!\Gamma(\alpha+\beta+Y+2)}\sum_{k=0}^{n}\frac{(-1)_K(1+\alpha+\beta+n)_K}{K(\alpha+\beta+Y+2)_K}$$

$$=\frac{2^{\alpha+\beta+Y+1}\Gamma(\alpha+n+1)\Gamma(\beta+Y+1)(-1)^n(-Y)_n}{n!\Gamma(\alpha+\beta+Y+2)(\alpha+\beta+Y+2)_n}$$

$$=\frac{2^{\alpha+\beta+Y+1}\Gamma(\alpha+n+1)\Gamma(\beta+Y+1)\Gamma(Y+1)}{n!\Gamma(\alpha+\beta+Y+n+2)\Gamma(Y-n+1)}$$

Hence the required result.

PROOF of (3).

Consider the integral

$$C=\int_{-\infty}^{\infty}e^{-x^2}x^{2l}H_{2n}(x)\ dx. \tag{3. 1}$$

From [1, p. 187(2)] and (3. 1), we have

$$C=\sum_{k=0}^{n}\frac{(-1)^K(2n)!2^{2n-2K}}{K!(2n-2K)!}\int_{-\infty}^{\infty}e^{-x^2}x^{2l+2n-2K}dx$$

$$=2^{2n}\Gamma(l+n+1/2)\sum_{k=0}^{n}\frac{(-n)_K(-n+1/2)_K}{K!(1/2-l-n)_K}.$$

If $0 \leq n \leq l$, then

$$C = 2^{2n} \frac{\Gamma(l+1)\Gamma(l+1/2)}{\Gamma(l-n+1)}.$$

Hence the precise result.

The proof of (4) is similar to that of (3).

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REFERENCE

[1] Earl D. Rainville. *Special Functions* (MacMillan), 1960.