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SOME CONTOUR INTEGRALS INVOLVING GENERALIZED LEGENDRE FUNCTIONS AND THE H-FUNCTION

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1. Introduction

The object of the present paper is to evaluate some contour integrals involving products of generalized Legendre functions and the H-function. In these results, integration is performed with respect to the parameters of generalized Legendre functions. Generalized Legendre functions reduce to associated Legendre functions on setting $m=n$ and to Legendre functions on taking $m=n=0$. Also on specializing the parameters of the H-function, which is a very general function, we get various other known functions. Thus, on specializing the parameters of these functions, we may get many interesting relations.

The H-function was introduced by Fox [3, p. 408] and its conditions of validity, asymptotic expansions and analytic continuations have been discussed by Braaksma [1]. Following the definition given by Braaksma [1, pp. 239—241], it will be represented as follows:

$$(1.1)\quad H_{r,s}^{l,u}\left[z\left|\begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_T\frac{\prod_{j=1}^{l}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{u}\Gamma(1-a_j+\alpha_j\xi)z^{\xi}}{\prod_{j=l+1}^{s}\Gamma(1-b_j+j)\prod_{j=u+1}^{r}\Gamma(a_j-\alpha_j\xi)}d\xi,$$

where $\{(a_r,\ \alpha_r)\}$, stands for the set of parameters $(a_1,\alpha_1),\ \cdots,\ (a_r,\alpha_r)$.

Generalized Legendre functions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$, two specified linearly independent solutions of the differential equation:

$$(1.2)\quad (1-z^2)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+\left\{k(k+1)-\frac{m^2}{2(1-z)}-\frac{n^2}{2(1+z)}\right\}w=0,$$

have been introduced by Kuipers and Meulenbeld [4] as functions of z for all points of the z-plane, in which a cross-cut exists along the real x-axis from 1 to $-\infty$. On the segment $-1<x<1$ of the cross-cut these functions are defined in [5, (1) and (2)].

One of the theorems of Braaksma-Meulenbeld [2] is:

Let n_1 be a real number with $n_1<\min\{\text{Re}(2k+2-m), \text{Re}(-2k-m)\}$, and $\phi(t)$ a function such that for all a, $-1<a<1$:

$$\phi(t)\ (1+t)^{-\frac{1}{4}-\frac{1}{2}|\text{Re}\, m|}\in L(-1,a)\quad \text{if Re } m\neq 0,$$

$$\phi(t)\ (1+t)^{-\frac{1}{4}}\log(1+t)\in L(-1,\ a)\quad \text{if Re } m=0,$$

$$\phi(t)\ (1-t)^{-1-\frac{1}{2}n_1}\in L(a,1).$$

Let further $\phi(t)$ be of bounded variation in a neighbourhood of $t=x(-1<x<1)$. Then $\phi(t)$ satisfies the relations:

$$(1.3)\quad \frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty}dn\ n\Gamma(\beta+1)\Gamma(-\alpha)P_k^{m,n}(-x)\int_{-1}^{1}\phi(t)P_k^{m,n}(t)\frac{dt}{1-t}$$
$$=-\{\phi(x-0)+\phi(x+0)\},$$

and

$$(1.4)\quad \frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty}dn\ n\Gamma(\beta+1)\Gamma(-\alpha)P_k^{n,m}(x)\int_{-1}^{1}\phi(t)P_k^{m,n}(-t)\frac{dt}{1-t}$$
$$=-\{\phi(x-0)+\phi(x+0)\},$$

where $\alpha=k+\frac{1}{2}(m+n)$ and $\beta=k-\frac{1}{2}(m+n)$.

In what follows, for the sake of brevity

$$\sum_1^s(\beta_j)-\sum_1^k(\alpha_j)\equiv A,\quad \sum_1^l(\beta_j)-\sum_{l+1}^s(\beta_j)+\sum_1^u(\alpha_j)-\sum_{u+1}^k(\alpha_j)\equiv B.$$

2. In this section, we establish an integral which is required in the development of the present work.

If $\delta>0$, $A\geq 0$, $B>0$, $|\arg z|<\frac{1}{2}B\pi$,

Re $(p-\frac{1}{2}n+\delta b_j/\beta_j)>-1$ $(j=1,2,\cdots,l)$ and

Re $(k-p+\delta(1-a_i)/\alpha_i)>\frac{1}{2}|\text{Re}\ m|$ $(i=1,2,\cdots,u)$, then we have:

$$(2.1)\quad \int_{-1}^{1}(1-t)^p(1+t)^{k-p-1}P_k^{n,m}(t)H_{r,s}^{l,u}\left[z\left(\frac{1-t}{1+t}\right)^{\delta}\middle|\begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right]dt$$
$$=2^{k+\frac{1}{2}(m-n)}\left[\Gamma\left(k+\frac{1}{2}(m-n)+1\right)\Gamma\left(k-\frac{1}{2}(m+n)+1\right)\right]^{-1}\times$$

$$\times H^{l+2,u+1}_{r+2,s+2}\left[z\left|\begin{matrix}\left(\frac{1}{2}n-p,\ \delta\right),\ \{(a_r,\alpha_r)\},\ \left(-p-\frac{1}{2}n,\ \delta\right)\\ \left(k-p+\frac{1}{2}m,\ \delta\right),\ \left(k-p-\frac{1}{2}m,\ \delta\right),\ \{(b_s,\ \beta_s)\}\end{matrix}\right.\right].$$

PROOF. To obtain the integral (2.1), expressing the H function as Mellin-Barnes type of integral (1.1) and interchanging the order of integration, which is justifiable due to the absolute convergence of the integrals involved in the process, we get

$$(2.2)\quad \frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{l}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{u}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=l+1}^{s}\Gamma(1-b_j+\beta_j\xi)\prod_{j=u+1}^{r}\Gamma(a_j-\alpha_j\xi)}z^{\xi}$$

$$\times\int_{-1}^{1}(1-t)^{p+\delta\xi}(1+t)^{k-p-1-\delta\xi}P_k^{n,m}(t)dt\ d\xi,$$

evaluating the inner integral with the help of [2, *p*. 277(8.2)], i.e.

$$(2.3)\quad \int_{-1}^{1}(1-t)^{p}(1+t)^{k-p-1}P_k^{n,m}(t)\ dt$$

$$=2^{k+\frac{1}{2}(m-n)}\frac{\Gamma\left(k-p+\frac{1}{2}m\right)\Gamma\left(k-p-\frac{1}{2}m\right)\Gamma\left(p-\frac{1}{2}n+1\right)}{\Gamma\left(k+\frac{1}{2}(m-n)+1\right)\Gamma\left(k-\frac{1}{2}(m+n)+1\right)\Gamma\left(p-\frac{1}{2}n\right)},$$

where Re $\left(p-\frac{1}{2}n\right)>-1$ and Re $(k-p)>\frac{1}{2}|\text{Re } m|$; and again applying (1.1), the definition of the H-function, we get the value of the integral.

3 (i) Let $\delta>0$, $A\geq 0$, $B>0$, $|\arg z|<\frac{1}{2}B\pi$, p be a complex number, and n_1 a real number with

$n_1<\min\{2+2\text{ Re }(p+\delta b_j/\beta_j),\ -\text{Re }(2k+m)\}$ $(j=1,\ 2,\ \cdots,\ l)$ and $\text{Re}(k-p+\delta(1-a_i)/\alpha_i)>\frac{1}{4}(i=1,\ 2,\ \cdots,\ u)$.

Then for $-1<x<1$, we have:

$$(3.1)\quad \frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty}2^{-\frac{1}{2}n}\ n\frac{\Gamma\left(-k-\frac{1}{2}(m+n)\right)}{\Gamma\left(k+\frac{1}{2}(m-n)+1\right)}P_k^{m,n}(x)$$

$$\times H^{l+2,\ u+1}_{r+2,\ s+2}\left[z\left|\begin{matrix}\left(\frac{1}{2}n-p,\ \delta\right),\ \{(a_r,\ \alpha_r)\},\ \left(-p-\frac{1}{2}n,\ \delta\right)\\ \left(k-p+\frac{1}{2}m,\ \delta\right),\ \left(k-p-\frac{1}{2}m,\ \delta\right),\ \{(b_s,\ \beta_s)\}\end{matrix}\right.\right]dn$$

$$=-2^{1-k-\frac{1}{2}m}(1+x)^{p+1}(1-x)^{k-p-1}H^{l,u}_{r,s}\left[z\left(\frac{1+x}{1-x}\right)^{\delta}\middle|\begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right].$$

PROOF. In (1.3), taking

$$\phi(t)=(1-t)^{p+1}(1+t)^{k-p-1}H^{l,\ u}_{r,\ s}\left[z\left(\frac{1-t}{1+t}\right)^{\delta}\middle|\begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right],$$

and applying (2.1), we obtain (3.1).

(ii) Now in (3.1), using the relation

$$(3.2)\quad H^{l,u}_{r,s}\left[z\middle|\begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right]=H^{u,l}_{s,r}\left[z^{-1}\middle|\begin{matrix}\{(1-b_s,\ \beta_s)\}\\ \{(1-a_r,\ \alpha_r)\}\end{matrix}\right],$$

then replacing $l,\ u,\ r,\ s,\ z^{-1},\ \{(1-b_s,\beta_s)\}$ and $\{(1-a_r,\alpha_r)\}$ respectively by $u,\ l,\ s,\ r,\ z,\ \{(a_r,\ \alpha_r)\}$ and $\{(b_s,\ \beta_s)\}$, we have

$$(3.3)\quad \frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty}2^{-\frac{1}{2}n}\,n\frac{\Gamma\left(-k-\frac{1}{2}(m+n)\right)}{\Gamma\left(k+\frac{1}{2}(m-n)+1\right)}\ P_k^{m,n}(x)$$

$$\times H^{l+1,\ u+2}_{r+2,\ s+2}\left[z\middle|\begin{matrix}\left(1-k+p+\frac{1}{2}m,\ \delta\right),\ \left(1-k+p+\frac{1}{2}m,\ \delta\right),\ \{(a_r,\ \alpha_r)\}\\ \left(1-\frac{1}{2}n+p,\ \delta\right),\ \{(b_s,\ \beta_s)\},\ \left(1+p+\frac{1}{2}n,\ \delta\right)\end{matrix}\right]dn$$

$$=-2^{1-k-\frac{1}{2}m}(1+x)^{p+1}(1-x)^{k-p-1}H^{l,u}_{r,s}\left[z\left(\frac{1-x}{1+x}\right)^{\delta}\middle|\begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\},\end{matrix}\right]$$

where $\delta>0, A\geq 0, B>0, |\arg z|<\frac{1}{2}B\pi,\ -1<x<1,\ n_1<\min\{2+2\,\mathrm{Re}(p+\delta(1-a_i)/\alpha_i),\ -\mathrm{Re}(2k+m)\}\,(i=1,\ 2,\ \cdots,\ u)$ and $\mathrm{Re}\ (k-p+\delta b_j/\beta_j)<\frac{1}{4}(j=1,\ 2,\ \cdots,\ l)$.

(iii) In (3.1) and (3.3), applying the relation [2, p. 239(1.1)], i.e.

$$(3.4)\quad P_k^{m,n}(x)=2^{k+\frac{1}{2}n+1}(1+x)^{-\frac{1}{2}}P^{m,\ -2k-1}_{-\frac{1}{2}(n+1)}\left(\frac{x-3}{-x-1}\right),\ -1<x<1\ ;$$

substituting $\frac{x-3}{-x-1}$ for x and replacing k by $-\frac{1}{2}(n+1)$, n by $(-2k-1)$ and n_1 by $-2k_1-1$, we obtain:

$$(3.5)\quad \frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}(2k+1)\frac{\Gamma\left(k-\frac{1}{2}(m-n)+1\right)}{\Gamma\left(k+\frac{1}{2}(m-n)+1\right)}\ P_k^{m,n}(x)$$

$$\times H^{l+2,n+1}_{r+2,s+2}\left[z\middle|\begin{matrix}\left(-k-p-\frac{1}{2},\ \delta\right),\ \{(a_r,\ \alpha_r)\},\ \left(k-p+\frac{1}{2},\ \delta\right)\\ \left(-p+\frac{1}{2}m-\frac{1}{2}n-\frac{1}{2},\ \delta\right),\ \left(-p-\frac{1}{2}m-\frac{1}{2}n-\frac{1}{2},\delta\right),\ \{(b_s,\ \beta_s)\}\end{matrix}\right]dk$$

$$=2^{p+\frac{1}{2}(n-m)+\frac{3}{2}}(x-1)^{-p-\frac{1}{2}n-\frac{3}{2}}(x+1)^{\frac{1}{2}n}\ H^{l,u}_{r,s}\left[\frac{2^{\delta}z}{(x-1)^{\delta}}\middle|\begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right],$$

provided $\delta>0$, $A\geq 0$, $B>0$, $|\arg z|<\frac{1}{2}B\pi$, $x>1$,

$$k_1>\max\left\{-\frac{3}{2}-\mathrm{Re}(p+\delta b_j/\beta_j),\ \frac{1}{2}\ \mathrm{Re}\ (m-n)-1\right\}\ (j=1,\ 2,\ \cdots,\ l)$$

and $\mathrm{Re}(n+2p-2\delta(1-a_i)/\alpha_i)<-\frac{3}{2}(i=1,\ 2,\ \cdots,\ u)$; and

$$(3.6)\quad \frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}(2k+1)\frac{\Gamma\left(k-\frac{1}{2}(m-n)+1\right)}{\Gamma\left(k+\frac{1}{2}(m-n)+1\right)}P_k^{m,n}(x)$$

$$\times H^{l+1,u+2}_{r+2,s+2}\left[z\ \middle|\begin{matrix}\left(\frac{3}{2}-\frac{1}{2}m+\frac{1}{2}n+p,\ \delta\right),\ \left(\frac{3}{2}+\frac{1}{2}m+\frac{1}{2}n+p,\delta\right),\ \{(a_r,\alpha_r)\}\\ \left(\frac{3}{2}+k+p,\ \delta\right),\ \{(b_s,\ \beta_s)\},\ \left(\frac{1}{2}-k+p,\ \delta\right)\end{matrix}\right]dk$$

$$=2^{p+\frac{1}{2}(n-m)+\frac{3}{2}}(x-1)^{-p-\frac{1}{2}n-\frac{3}{2}}(x+1)^{\frac{1}{2}n}\ H^{l,\ u}_{r,\ s}\left[z\left(\frac{x-1}{2}\right)^{\delta}\ \middle|\ \begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right],$$

where $\delta>0$, $A\geq 0$, $B>0$, $|\arg z|<\frac{1}{2}B\pi$, $x>1$,

$$k_1>\max\left\{-\frac{3}{2}-\mathrm{Re}(p+\delta(1-a_i)/\alpha_i),\ \frac{1}{2}\mathrm{Re}(m-n)-1\right\}\ (i=1,\ 2,\ \cdots,\ u)$$

and $\mathrm{Re}(n+2p-2\delta b_j/\beta_j)<-\frac{3}{2}\ (j=1,\ 2,\ \cdots,\ l)$.

(iv) Taking $k=-\frac{1}{2}m-\frac{1}{2}$ in (3.1) and (3.3), we obtain:

$$(3.7)\quad \frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty}2^{-\frac{1}{2}n}n\,P^{m,n}_{-\frac{1}{2}(m+1)}(x)$$

$$\times H^{l+2,u+1}_{r+2,s+2}\left[z\ \middle|\begin{matrix}\left(\frac{1}{2}n-p,\ \delta\right),\ \{(a_r,\ \alpha_r)\},\ \left(-p-\frac{1}{2}n,\ \delta\right)\\ \left(p-\frac{1}{2},\ \delta\right),\ \left(-p-m-\frac{1}{2},\ \delta\right),\ \{(b_s,\ \beta_s)\}\end{matrix}\right]dn$$

$$=-2^{\frac{3}{2}}(1+x)^{p+1}(1-x)^{-p-\frac{1}{2}m-\frac{3}{2}}H^{l,u}_{r,s}\left[z\left(\frac{1+x}{1-x}\right)^{\delta}\ \middle|\ \begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right],$$

where $\delta>0$, $A\geq 0$, $B>0$, $|\arg z|<\frac{1}{2}B\pi$, $-1<x<1$,

$n_1<\min\{2+2\ \mathrm{Re}(p+\delta b_j/\beta_j),\ 1\}\ (j=1,\ 2,\ \cdots,\ l)$ and

$\mathrm{Re}\left(p+\frac{1}{2}m+\delta(a_i-1)/\alpha_i\right)<-\frac{3}{4}\ (i=1,\ 2,\ \cdots,\ u)$;

and

$$(3.8)\quad \frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty} 2^{-\frac{1}{2}n} n P^{m,n}_{-\frac{1}{2}(m+1)}(x)$$

$$\times H^{l+1,\,u+2}_{r+2,\,s+2}\left[z\left|\begin{matrix}\left(\frac{3}{2}+p,\ \delta\right),\ \left(\frac{3}{2}+m+p,\ \delta\right),\ \{(a_r,\ \alpha_r)\}\\ \left(1-\frac{1}{2}n+p,\ \delta\right),\ \{(b_s,\ \beta_s)\},\ \left(1+p+\frac{1}{2}n,\ \delta\right)\end{matrix}\right.\right]dn$$

$$=-2^{\frac{3}{2}}(1+x)^{p+1}(1-x)^{-p-\frac{1}{2}m-\frac{3}{2}}H^{l,u}_{r,s}\left[z\left(\frac{1-x}{1+x}\right)^{\delta}\left|\begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right.\right],$$

provided $\delta>0$, $A\geq 0$, $B>0$, $|\arg z|<\frac{1}{2}B\pi$, $-1<x<1$, $n_1<\min\{2+2\mathrm{Re}(p+\delta(1-a_i)/\alpha_i),\ 1\}$ $(i=1,\ 2,\ \cdots,\ u)$ and $\mathrm{Re}\left(p+\frac{m}{2}-b_j/\beta_j\right)<-\frac{3}{4}$ $(j=1,\ 2,\ \cdots,\ l)$.

(v) Putting $n=m$ in (3.5) and (3.6), we have

$$(3.9)\quad \frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}(2k+1)\,P^{m}_{k}(x)$$

$$\times H^{l+2,u+1}_{r+2,s+2}\left[z\left|\begin{matrix}\left(-k-p-\frac{1}{2},\ \delta\right),\ \{(a_r,\ \alpha_r)\},\ \left(k-p+\frac{1}{2},\ \delta\right)\\ \left(-p-\frac{1}{2},\ \delta\right),\ \left(-p-m-\frac{1}{2},\ \delta\right),\ \{(b_s,\ \beta_s)\}\end{matrix}\right.\right]dk$$

$$=2^{p+\frac{3}{2}}(x-1)^{-p-\frac{1}{2}m-\frac{3}{2}}(x+1)^{\frac{1}{2}m}H^{l,u}_{r,s}\left[\frac{2^{\delta}z}{(x-1)^{\delta}}\left|\begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right.\right],$$

if $\delta>0$, $A\geq 0$, $B>0$, $|\arg z|<\frac{1}{2}B\pi$, $x>1$,

$k_1>\max\left\{-\frac{3}{2}-\mathrm{Re}(p+\delta b_j/\beta_j),\ -\frac{1}{2}\right\}$ $(j=1,\ 2,\ \cdots,\ l)$ and

$\mathrm{Re}(m+2p+2\delta(a_i-1)/\alpha_i)<-\frac{3}{2}$ $(i=1,\ 2,\ \cdots,\ u)$;

and

$$(3.10)\quad \frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}(2k+1)P^{m}_{k}\ (x)$$

$$\times H^{l+1,\,u+2}_{r+2,\,s+2}\left[z\left|\begin{matrix}\left(\frac{3}{2}+p,\ \delta\right),\ \left(\frac{3}{2}+p+m,\ \delta\right),\ \{(a_r,\ \alpha_r)\}\\ \left(\frac{3}{2}+k+p,\ \delta\right),\ \{(b_s,\ \beta_s)\},\ \left(\frac{1}{2}-k+p,\ \delta\right)\end{matrix}\right.\right]dk$$

$$=2^{p+\frac{3}{2}}(x-1)^{-p-\frac{1}{2}m-\frac{3}{2}}(x+1)^{\frac{1}{2}m}H^{l,u}_{r,s}\left[z\left(\frac{x-1}{2}\right)^{\delta}\left|\begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right.\right],$$

where $\delta>0$, $A\geq 0$, $B>0$, $|\arg z|<\frac{1}{2}B\pi$, $x>1$,

$k_1>\max\left\{-\frac{3}{2}-\mathrm{Re}(p+\delta(1-a_i)/\alpha_i),\ -\frac{1}{2}\right\}$ $(i=1,\ 2,\ \cdots,\ u)$

and $\mathrm{Re}(m+2p-2\delta b_j/\beta_j)<-\frac{3}{2}$ $(j=1, 2, \cdots, l)$.

(vi) By applying [2, p. 235(0.2)], i.e.

$$(3.11)\quad P_k^{m,n}(z)=\frac{(z-1)^{-\frac{1}{2}m}(z+1)^{\frac{1}{2}n}}{\Gamma(1-m)}\times{}_2F_1\left[\begin{matrix}1+k-\frac{1}{2}(m-n);\ -k-\frac{1}{2}(m-n);\ \frac{1-z}{2}\\ 1-m\end{matrix}\right],$$

in (3.5) and (3.6) and then replacing $\frac{1}{2}(n-m)$ by a, $1-m$ by m and $1-x$ by $-2x$, we get

$$(3.12)\quad \frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}(2k+1)\frac{\Gamma(k+a+1)}{\Gamma(k-a+1)}{}_2F_1(-k+a,\ k+a+1;\ m;-x)$$
$$\times H_{r+2,s+2}^{l+2,u+1}\left[z\left|\begin{matrix}\left(-k-p-\frac{1}{2},\ \delta\right),\ \{(a_r,\ \alpha_r)\},\ \left(k-p+\frac{1}{2},\ \delta\right)\\ \left(-p-a-\frac{1}{2},\ \delta\right),\ \left(m-p-a-\frac{3}{2},\ \delta\right),\ \{(b_s,\ \beta_s)\}\end{matrix}\right.\right]dk$$
$$=x^{-p-a-\frac{3}{2}}\Gamma(m)\ H_{r,s}^{l,u}\left[\frac{z}{x^\delta}\left|\begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right.\right],$$

provided $\delta>0$, $A\geq 0$, $B>0$, $|\arg z|<\frac{1}{2}B\pi$, $x>0$,

$k_1>\max\left\{-\frac{3}{2}-\mathrm{Re}(p+\delta b_j/\beta_j),\ -\mathrm{Re}\ a-1\right\}$ $(j=1, 2, \cdots, l)$

and $\mathrm{Re}\ (2p+2a-m+2\delta(a_i-1)/\alpha_i)<-\frac{5}{2}$ $(i=1, 2, \cdots, u)$;

and

$$(3.13)\quad \frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}(2k+1)\frac{\Gamma(k+a+1)}{\Gamma(k-a+1)}{}_2F_1(-k+a,\ k+a+1;\ m;\ -x)$$
$$\times H_{r+2,s+2}^{l+1,u+2}\left[z\left|\begin{matrix}\left(\frac{3}{2}+a+p,\ \delta\right),\ \left(\frac{5}{2}+a+p-m,\ \delta\right),\ \{(a_r,\ \alpha_r)\}\\ \left(\frac{3}{2}+k+p,\ \delta\right),\ \{(b_s,\ \beta_s)\},\ \left(\frac{1}{2}-k+p,\ \delta\right)\end{matrix}\right.\right]dk$$
$$=x^{-p-a-\frac{3}{2}}\Gamma(m)\ H_{r,s}^{l,u}\left[zx^\delta\left|\begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right.\right],$$

if $\delta>0$, $A\geq 0$, $B>0$, $|\arg z|<\frac{1}{2}B\pi$, $x>0$,

$k_1>\max\left\{-\frac{3}{2}-\mathrm{Re}(p+\delta(1-a_i)/\alpha_i),\ -\mathrm{Re}\ a-1\right\}$ $(i=1, 2, \cdots, u)$

and $\mathrm{Re}\ (2p+2a-m-2\delta b_j/\beta_j)<-\frac{5}{2}$ $(j=1, 2, \cdots, l)$.

(vii) Taking $a=0$ in (3.12) and (3.13), we obtain:

$$(3.14)\quad \frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}(2k+1)\ {}_2F_1(-k,\ k+1;\ m;\ -x)$$

$$\times H^{l+2,u+1}_{r+2,s+2}\left[z\ \middle|\ \begin{matrix}\left(-k-p-\frac{1}{2},\ \delta\right),\ \{(a_r,\ \alpha_r)\},\ \left(k-p+\frac{1}{2},\ \delta\right)\\ \left(-p-\frac{1}{2},\ \delta\right),\ \left(m-p-\frac{3}{2},\ \delta\right),\ \{(b_s,\ \beta_s)\}\end{matrix}\right]dk$$

$$=x^{-p-\frac{3}{2}}\Gamma(m)\ H^{l,u}_{r,s}\left[\frac{z}{x^{\delta}}\ \middle|\ \begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right],$$

where $\delta>0$, $A\geq 0$, $B>0$, $|\arg z|<\frac{1}{2}B\pi$, $x>0$,

$k_1>\max\left\{-\frac{3}{2}-\mathrm{Re}(p+\delta b_j/\beta_j),\ -1\right\}$ $(j=1,\ 2,\ \cdots,\ l)$ and

$\mathrm{Re}(2p-m+2\delta(a_i-1)/\alpha_i)<-\frac{5}{2}$ $(i=1,\ 2,\ \cdots,\ u)$;

and

$$(3.15)\quad \frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}(2k+1)\ {}_2F_1(-k,\ k+1;\ -x)$$

$$\times H^{l+1,u+2}_{r+2,s+2}\left[z\ \middle|\ \begin{matrix}\left(\frac{3}{2}+p,\ \delta\right),\ \left(\frac{5}{2}+p-m,\ \delta\right),\ \{(a_r,\ \alpha_r)\}\\ \left(\frac{3}{2}+k+p,\ \delta\right),\ \{(b_s,\ \beta_s)\},\ \left(\frac{1}{2}-k+p,\ \delta\right)\end{matrix}\right]dk$$

$$=x^{-p-\frac{3}{2}}\Gamma(m)\ H^{l,u}_{r,s}\left[zx^{\delta}\ \middle|\ \begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right],$$

Provided $\delta>0$, $A\geq 0$, $B>0$, $|\arg z|<\frac{1}{2}B\pi$, $x>0$,

$k_1>\max\left\{-\frac{3}{2}-\mathrm{Re}(p+\delta(1-a_i)/\alpha_i),\ -1\right\}$ $(i=1,\ 2,\ \cdots,\ u)$ and

$\mathrm{Re}(2p-m-2\delta b_j/\beta_j)<-\frac{5}{2}$ $(j=1,\ 2,\ \cdots,\ l)$.

(viii) In (3.5) and (3.6), we suppose $m-n\neq 1,\ 2,\ \cdots$ and deform the path of integration such that the points $k=\frac{1}{2}(m-n)-g$ $(g=1,\ 2,\ \cdots)$and $k=-\frac{1}{2},\ 1,\ -1\frac{1}{2},\ -2,\cdots$ are to the left, and the points $k=-\frac{1}{2}(m-n)+h$ $(h=0,\ 1,\ \cdots)$ and $k=0,\ \frac{1}{2},\ 1,\ 1\frac{1}{2},\ \cdots$ are to the right of the new path L and replacing δ by d and using [2, p. 246(1.39)}, viz.

$$(3.16)\quad e^{\pi i m}\left\{Q^{-m,-n}_{\ \ k}(t)-Q^{-m,-n}_{-k-1}(t)\right\}$$

$$=2^{m-n-1}\Gamma(\beta+1)\Gamma(\delta+1)\ \Gamma(-\alpha)\Gamma(-\gamma)\frac{\sin(2k\pi)}{\pi}\ P^{m,n}_{k}(t),$$

where $\alpha=k+\frac{1}{2}(m+n)$; $\beta=k-\frac{1}{2}(m-n)$, $\gamma=k+\frac{1}{2}(m-n)$ and $\delta=k-\frac{1}{2}(m+n)$; we get

$$(3.17)\quad \frac{1}{2\pi i}\int_L \frac{(2k+1)}{\sin(2k\pi)}\Big[\Gamma(\delta+1)\Gamma(-\alpha)\Gamma(-\gamma)\Gamma(\gamma+1)\Big]^{-1} \times\Big\{Q_k^{-m,-n}(x)-Q_{-k-1}^{-m,-n}(x)\Big\}$$
$$\times H_{r+2,s+2}^{l+2,u+1}\left[z\,\middle|\,\begin{matrix}\left(-k-p-\frac{1}{2},\ d\right),\ \{(a_r,\ \alpha_r)\},\ \left(k-p+\frac{1}{2},\ d\right)\\ \left(-p+\frac{1}{2}m-\frac{1}{2}n-\frac{1}{2},\ d\right),\ \left(-p-\frac{1}{2}m-\frac{1}{2}n-\frac{1}{2},d\right),\ \{(b_s,\ \beta_s)\}\end{matrix}\right]dk$$
$$=2^{p+\frac{1}{2}(m-n)+\frac{1}{2}}\pi^{-1}e^{-\pi im}(x-1)^{-p-\frac{1}{2}n-\frac{3}{2}}(x+1)^{\frac{1}{2}n} \times H_{r,s}^{l,u}\left[\frac{2^d z}{(x-1)^d}\,\middle|\,\begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right],$$

where $d>0$, $A\geq 0$, $B>0$, $|\arg z|<\frac{1}{2}B\pi$, $x>1$; α, β,γ and δ are as given in (3.16) ; $\mathrm{Re}\left(k+\frac{3}{2}+p+db_j/\beta_j\right)>0$ $(j=1,\ 2,\ \cdots,\ l)$ and $\mathrm{Re}(n+2p+2d(a_i-1)/\alpha_i)<-\frac{3}{2}(i=1,\ 2,\ \cdots,\ u)$; and

$$(3.18)\quad \frac{1}{2\pi i}\int_L \frac{(2k+1)}{\sin(2k\pi)}\Big[\Gamma(\delta+1)\Gamma(-\alpha)\Gamma(-\gamma)\Gamma(\gamma+1)\Big]^{-1} \times\Big\{Q_k^{-m,-n}(x)-Q_{-k-1}^{-m,-n}(x)\Big\}$$
$$\times H_{r+2,s+2}^{l+1,u+2}\left[z\,\middle|\,\begin{matrix}\left(\frac{3}{2}-\frac{1}{2}m+\frac{1}{2}n+p,d\right),\left(\frac{3}{2}+\frac{1}{2}m+\frac{1}{2}n+p,d\right),\ \{(a_r,\alpha_r)\}\\ \left(\frac{3}{2}+k+p,d\right),\ \{(b_s,\beta_s)\},\left(\frac{1}{2}-k+p,d\right)\end{matrix}\right]dk$$
$$=2^{p+\frac{1}{2}(m-n)+\frac{1}{2}}\pi^{-1}e^{-\pi im}(x-1)^{-p-\frac{1}{2}n-\frac{3}{2}}(x+1)^{\frac{1}{2}n} \times H_{r,s}^{l,u}\left[z\left(\frac{x-1}{2}\right)^d\,\middle|\,\begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right],$$

where $d>0$, $A\geq 0$, $B>0$, $|\arg z|<\frac{1}{2}B\pi$, $x>1$, $\mathrm{Re}\left(k+\frac{3}{2}+p+d\,(1-a_i)/\alpha_i\right)>0$ $(i=1,\ 2,\ \cdots,\ u)$, $\mathrm{Re}(n+2p-2db_j/\beta_j)<-\frac{3}{2}(j=1,\ 2,\ \cdots,\ l)$ and α, β, γ, δ are as given in (3.16).

(ix) Let $\delta>0$, $A\geq 0$, $B>0$, $|\arg z|<\frac{1}{2}B\pi$, p be a complex number and n_1 a real number with $|n_1|<2\ \mathrm{Re}(k-p+\delta(1-a_i)/\alpha_i)(i=1,\ 2,\ \cdots,\ u)$, $n_1<-\mathrm{Re}(2k+m)$, $\mathrm{Re}(p+\delta b_j/\beta_j)>-\frac{1}{4}(j=1,\ 2,\ \cdots,\ l)$.

Then for $-1<x<1$, we have

$$(3.19)\quad \frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty} 2^{\frac{1}{2}n} n \frac{\Gamma\left(-k-\frac{1}{2}(m+n)\right)}{\Gamma\left(k-\frac{1}{2}(m-n)+1\right)} P_k^{n,m}(x)$$

$$\times H_{r+2,s+2}^{l+2,u+1}\left[z\left|\begin{matrix}\left(\frac{1}{2}m-p,\ \delta\right),\ \{(a_r,\ \alpha_r)\},\ \left(-p-\frac{1}{2}m,\ \delta\right)\\ \left(k-p+\frac{1}{2}n,\ \delta\right),\ \left(k-p-\frac{1}{2}n,\ \delta\right),\ \{(b_s,\ \beta_s)\}\end{matrix}\right.\right]dn$$

$$=-2^{1-k+\frac{1}{2}m}(1+x)^p(1-x)^{k-p}H_{r,s}^{l,u}\left[z\left(\frac{1+x}{1-x}\right)^{\delta}\left|\begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right.\right].$$

PROOF. In (1.4), taking

$$\phi(t)=(1+t)^p(1-t)^{k-p}H_{r,s}^{l,u}\left[z\left(\frac{1+t}{1-t}\right)^{\delta}\left|\begin{matrix}(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right.\right]$$

and applying (2.1) with t replaced by $-t$ and interchanging m and n, we can easily get (3.19).

(x) Now, using the relation (3.2) in (3.19) and then replacing l, u, r, s, z^{-1}, $\{(1-b_s,\ \beta_s)\}$ and $\{(1-a_r,\ \alpha_r)\}$ respectively by u, l, s, r, z, $\{(a_r,\ \alpha_r)\}$ and $\{(b_s,\ \beta_s)\}$, we have

$$(3.20)\quad \frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty} 2^{\frac{1}{2}n} n \frac{\Gamma\left(-k-\frac{1}{2}(m+n)\right)}{\Gamma\left(k-\frac{1}{2}(m-n)+1\right)} P_k^{n,m}(x)$$

$$\times H_{r+2,s+2}^{l+1,u+2}\left[z\left|\begin{matrix}\left(1-k+p-\frac{1}{2}n,\delta\right),\ \left(1-k+p+\frac{1}{2}n,\delta\right),\ \{(a_r,\ \alpha_r)\}\\ \left(1+p-\frac{1}{2}m,\delta\right),\ \{(b_s,\ \beta_s)\},\ \left(1+p+\frac{1}{2}m,\ \delta\right)\end{matrix}\right.\right]dn$$

$$=-2^{1-k+\frac{1}{2}m}(1+x)^p(1-x)^{k-p}H_{r,s}^{l,u}\left[z\left(\frac{1-x}{1+x}\right)^{\delta}\left|\begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right.\right],$$

where $\delta>0$, $A\geq 0$, $B>0$, $|\arg z|<\frac{1}{2}B\pi$, $|n_1|<2\ \mathrm{Re}(k-p+\delta b_j/\beta_j)$ $(j=1,2,\cdots,l)$, $n_1<-\mathrm{Re}(2k+m)$, $\mathrm{Re}(p+\delta(1-a_i)/\alpha_i)>-\frac{1}{4}(i=1,\ 2,\ \cdots,\ u)$ and $-1<x<1$.

(xi) Applying [2, p. 239(1.2)], *i.e.*

$$(3.21)\quad P_k^{r,n}(x)=\frac{2^{-k-\frac{1}{2}m+1}(1-x)^{-\frac{1}{2}}e^{\pi i n}}{\Gamma\left(-k-\frac{1}{2}(m+n)\right)\Gamma\left(k-\frac{1}{2}(m+n)+1\right)}Q_{-\frac{1}{2}(m+1)}^{-n,2k+1}\left(\frac{-x-3}{x-1}\right),$$

in (3.19) and (3.20), substituting $\frac{-x-3}{x-1}$ for x and replacing k by $\frac{1}{2}(n-1)$,

n by $-(2k+1)$ and n_1 by $-(2k_1+1)$, we get

$$(3.22)\quad \frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}(2k+1)\frac{Q_k^{m,n}(x)}{\Gamma\left(k+\frac{1}{2}(m+n)+1\right)\Gamma\left(-k+\frac{1}{2}(m+n)\right)}$$

$$\times H_{r+2,s+2}^{l+2,u+1}\left[z\left|\begin{matrix}\left(-\frac{1}{2}m-p,\ \delta\right),\ (a_r,\ \alpha_r),\ \left(-p+\frac{1}{2}m,\ \delta\right)\\ \left(\frac{1}{2}n-k-p-1,\delta\right),\ \left(k-p+\frac{1}{2}n,\delta\right),\ \{(b_s,\ \beta_s)\}\end{matrix}\right.\right]dk$$

$$=2^{-\frac{1}{2}m+n-p-1}e^{-\pi im}(x-1)^p(x+1)^{-\frac{1}{2}n}H_{r,s}^{l,u}\left[z\left(\frac{x-1}{2}\right)^\delta\left|\begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right.\right]$$

provided $\delta>0,\ A\geq 0,\ B>0,\ |\arg z|<\frac{1}{2}B\pi,\ x>1,$
$k_1>\frac{1}{2}\mathrm{Re}(n-m)-1,\ \mathrm{Re}(p+\delta b_j/\beta_j)>-\frac{1}{4}(j=1,\ 2,\ \cdots,\ l)$ and
$|2k_1+1|<\mathrm{Re}(n-1-2p+2\delta(1-a_i)/\alpha_i)\ (i=1,\ 2,\ \cdots,\ u);$
and

$$(3.23)\quad \frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}(2k+1)\frac{Q_k^{m,n}(x)}{\Gamma\left(k+\frac{1}{2}(m+n)+1\right)\Gamma\left(-k+\frac{1}{2}(m+n)\right)}$$

$$\times H_{r+2,s+2}^{l+1,u+2}\left[z\left|\begin{matrix}\left(2+p+k-\frac{1}{2}n,\delta\right),\ \left(1-k+p-\frac{1}{2}n,\delta\right),\ \{(a_r,\alpha_r)\}\\ \left(1+\frac{1}{2}m+p,\ \delta\right),\ \{(b_s,\ \beta_s)\},\ \left(1+p-\frac{1}{2}m,\delta\right)\end{matrix}\right.\right]dk$$

$$=2^{-\frac{1}{2}m+n-p-1}e^{-\pi im}(x-1)^p(x+1)^{-\frac{1}{2}n}H_{r,s}^{l,u}\left[\frac{2^\delta z}{(x-1)^\delta}\left|\begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right.\right],$$

where $\delta>0,\ A\geq 0,\ B>0,\ |\arg z|<\frac{1}{2}B\pi,\ x>1,$
$k_1>\frac{1}{2}\mathrm{Re}(n-m)-1,\ \mathrm{Re}(p+\delta(1-a_i)/\alpha_i)>-\frac{1}{4}(i=1,\ 2,\ \cdots,\ u)$
and $|2k_1+1|<\mathrm{Re}(n-1-2p+2\delta b_j/\beta_j)\ (j=1,\ 2,\ \cdots,\ l).$

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REFERENCES

[1] Braaksma, B. L. J., *Asymptotic expansions and analytic continuations for a class of Barnes-integrals*, Compos. Math., 15(1963), pp. 239—341.

[2] Braaksma, B. L. J. & Meulenbeld, B., *Integral Transforms with generalized Legendre Functions as kernels*, Compos. Math. 18(1967), pp. 235—287.

[3] Fox, C., *The G- and H-functions as symmetrical Fourier kernels*, Trans. Amer. Math. Soc., 3, 98(1961), pp. 395—429.

[4] Kuipers, L. & Meulenbeld, B., *On a generalization of Legendre associated differential equation I and II*, Nederl. Akad. Wetensch. Proc. Ser. A, 60(1957), pp. 436—450.

[5] Meulenbeld, B. ; *Generalized Legendre's associated functions for real values of the argument numerically less than unity*, Nederl. Akad. Wetensch. Proc. Ser. A, 61 (1958), pp. 557—563.