Kyungpook Math. J.
Volume 12, Number 1
June, 1972

ON THE SOLUTION OF AN INTEGRAL EQUATION INVOLVING A KERNEL OF MELLIN-BARNES TYPE INTEGRAL

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1. Introduction.

The object of the present paper is to obtain the solution of an integral equation whose kernel $S_{p,q,r}(x)$ has a Mellin-Barnes type integral representation. As the kernel used here is of general character, various integral equations involving Whittaker functions, Bessel functions, Meijer's G-function, Fox's H-Function etc. as kernels can be derived as particular cases. By the application of certain operators of fractional integration, the kernel has been reduced to an exponential function and consequently the transform will reduce to Laplace transform which can be inverted by well known results.

2. Integral equation.

We consider an integral equation over the interval $(0, \infty)$ as,

$$g(x)=\int_0^\infty S_{p,q,r}(xu)h(u)du \tag{1}$$

where

$$S_{p,q,r}(x)=\frac{1}{2\pi i}\int_c P(s)x^{-s}ds \tag{2}$$

and

$$P(s)=\frac{\prod_{j=1}^{p}\sqrt{\left(\frac{a_j+As}{p}\right)}\prod_{j=1}^{q}\sqrt{\left(\frac{b_j+B_js}{m_j}\right)}\prod_{j=1}^{r}\sqrt{\left(\frac{1+d_j-D_js}{n_j}\right)}}{\prod_{j=1}^{q}\sqrt{\left(\frac{c_j+B_js}{m_j}\right)}\prod_{j=1}^{r}\sqrt{\left(\frac{1+e_j-D_js}{n_j}\right)}} \tag{3}$$

The following are the conditions of validity of (1):

(i) $h(x)\in L_2(0,\infty)$;

(ii) $x>0$;

(iii) p and r are positive integers and q is a non negative integer;

(iv) $m_j>0$ for $j=1,\cdots,q$.

(v) $n_j>0$ for $j=1,\cdots,r$.

(vi) the contour c is a straight line parallel to the imaginary axis in the complex s-plane given by $s=\frac{1}{2}+it$, t being real and $-\infty<t<\infty$ and all the poles of $\sqrt{\left(\frac{a_j+As}{p}\right)}$ for $j=1,\cdots,p$ and $\sqrt{\left(\frac{b_j+B_js}{m_j}\right)}$ for $j=1,\cdots,q$ must lie to the left of c while those of $\sqrt{\left(\frac{1+d_j-D_js}{n_j}\right)}$ for $j=1,\cdots,r$ to the right of it;

(vii) $a_i\neq a_j$, $i\neq j$, $i=1,\cdots,p$. Similar conditions hold for all b_j and c_j, $j=1,\cdots,q$ and d_j and e_j, $j=1,\cdots,r$.

The function $S_{p,q,r}$ will be represented as,

$$S_{p,q,r}\left(x\,\middle|\,\begin{matrix}(c_1,B_1),\cdots,(c_q,B_q);(e_1,D_1),\cdots,(e_r,D_r)\\(a_1,A),\cdots,(a_p,A);(b_1,B_1),\cdots,(b_q,B_q);(d_1,D_1),\cdots,(d_r,D_r)\end{matrix}\right) \tag{4}$$

In what follows for the sake of brevity (a_p, A_p) represent the set of parameters $(a_1,A_1),\cdots,(a_p,A_p)$.

3. The mellin transform.

The mellin transform of $h(x)$ will be denoted by $m\{h(x)\}$. If $m\{h(x)\}=H(s)$ then we shall also use the symbolic expression $h(x)=m^{-1}\{H(s)\}$ where m^{-1} indicates the inverse mellin transform.

Formally we have

$$H(s)=m\{h(x)\}=\int_0^\infty x^{s-1}h(x)dx \tag{5}$$

$$h(x)=m^{-1}\{H(s)\}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}H(s)x^{-s}ds. \tag{6}$$

The simplest conditions are given in L_2-space, which we shall use here. If $f(x)\in L_2(0,\infty)$ and the l.i.m. is with index 2 then

$$F(s)=m\{f(x)\}=\underset{N\to\infty}{\text{l.i.m.}}\int_{\frac{1}{N}}^{N}f(z)z^{s-1}dz \tag{7}$$

and also

$$F(s)\in L_2\left(\frac{1}{2}-i\infty,\ \frac{1}{2}+i\infty\right) \tag{8}$$

If

$$F(s)\in L_2\left(\frac{1}{2}-i\infty,\ \frac{1}{2}+i\infty\right)$$

then

$$f(x)=m^{-1}\{F(s)\}=\frac{1}{2\pi i}\ \underset{N\to\infty}{\text{l. i. m.}}\int_{\frac{1}{2}-iN}^{\frac{1}{2}+iN}F(s)x^{-s}ds \tag{9}$$

and also $f(x)\in L_2(0,\infty)$ [12, p. 94].

4. Fractional integration.

Several definitions of fractional integration have been given from time to time by many authors including Kober [9], Erdélyi [2], Saxena [11], Kalla and Saxena [6] and Kalla [7, 8]. In our present investigation we shall require the following operators of fractional integration defined by Erdélyi [2].

$$R(\alpha,\beta:m)h(x)=\frac{m}{\sqrt{\alpha}}x^{\beta}\int_x^{\infty}t^{-\beta-m\alpha+m-1}(t^m-x^m)^{\alpha-1}h(t)dt \tag{10}$$

and

$$S(\alpha,\beta:m)h(x)=\frac{m}{\sqrt{\alpha}}x^{-\beta-m\alpha+m-1}\int_0^{x}(x^m-t^m)^{\alpha-1}t^{\beta}h(t)dt \tag{11}$$

The conditions of validity of (10) and (11) are $\alpha>0$, $\beta>-\frac{1}{2}$, $m>0$ and $h(x)\in L_2(0,\infty)$.

Under these conditions $R(\alpha,\beta:m)h(x)$ and $S(\alpha,\beta:m)h(x)$ both belong to $L_2(0,\infty)$.

The mellin transform of these operators given by Erdélyi [2] are as follows:

$$m\{R(\alpha,\beta:m)h(x)\}=\frac{\sqrt{\left(\frac{\beta+s}{m}\right)}}{\sqrt{\left(\alpha+\frac{\beta+s}{m}\right)}}m\{h(x)\}, \tag{12}$$

$$m\{s(\alpha,\beta:m)h(x)\}=\frac{\sqrt{\left(\frac{\beta+1-s}{m}\right)}}{\sqrt{\left(\alpha+\frac{\beta+1-s}{m}\right)}}m\{h(x)\}. \tag{13}$$

5. Preliminary lemmas.

LEMMA 1. *If* (i) $x>0$, (ii) $h(t)$ *and* $g(t)$ *both belong to* $L_2(0,\infty)$, (iii) $m\{h(t)\}=H(s)$, $m\{g(t)\}=G(s)$ *and* $G(s)$ *is bounded on the line* $s=\frac{1}{2}+it$, $-\infty<t<\infty$, *then*

$$\int_0^{\infty}g(xt)h(t)dt\in L_2(0,\infty) \tag{14}$$

and

$$m\{\int_0^{\infty}g(xt)h(t)dt\}=G(s)H(1-s) \tag{15}$$

where the integrals of (14) and (15) are regarded as functions of x. This result is due to Fox [4, p. 458].

LEMMA 2. *If* (i) $x>0$, (ii) $a_j\geq 0$, $j=1,\cdots,p$ (iii) $b_j\geq 0, j=1,\cdots,q$ (iv) $d_j\geq 0$, $j=1,\cdots,r$ (v) $h(x)\in L_2(0,\infty)$, *then*

$$\int_0^\infty S_{p,q,r}(xt)\ h(t)\ dt\in L_2(0,\infty) \tag{16}$$

and

$$m\{\int_0^\infty S_{p,q,r}(xt)\ h(t)\ dt\}=P(s)H(1-s) \tag{17}$$

PROOF. We shall prove that

(a) $S_{p,q,r}(t)\in L_2(0,\infty)$

(b) $m\{S_{p,q,r}(t)\}$ is bounded on the line $s=\frac{1}{2}+it$.

From the asymptotic expansion of Gamma function [14, p. 279] we see that along the line $s=\frac{1}{2}+it$, $-\infty<t<\infty$ for large positive and negative t

$$|P(s)|=\lambda|t|^\mu \exp\left(-\frac{1}{2}\alpha\pi|t|\right) \tag{18}$$

where λ is a constant and

$$\mu=\frac{1}{p}\sum_{j=1}^{p}a_j+\sum_{j=1}^{q}\left(\frac{b_j-c_j}{m_j}\right)+\sum_{j=1}^{r}\left(\frac{d_j-e_j}{n_j}\right)-\frac{1}{2}(p+q)+\frac{A}{2p}$$

$$\alpha=\frac{A}{p}$$

Hence it follows that $P(s)$ is bounded on the line $s=\frac{1}{2}+it$ for all values of t and it belongs to $L_2\left(\frac{1}{2}-i\infty,\frac{1}{2}+i\infty\right)$. From (9) it follows that $S_{p,q,r}$ belongs to L_2 $(0,\infty)$ and consequently

$$m\{S_{p,q,r}(t)\}=P(s)$$

LEMMA 3. *If* $x>0$, $h(x)\in L_2(0,\infty), \alpha\geq 0$, $c_j>-\frac{1}{2}$, $j=1,\cdots,q$, *then*

$$R\left(\frac{\alpha}{mq},\frac{c_q}{B_q}:\frac{m_q}{B_q}\right)\int_0^\infty S_{p,q,r}(xu)h(u)du$$
$$=\int_0^\infty S_{p,q,r}\left(xu\,\middle|\,\begin{matrix}(c_{q-1},B_{q-1}):(c_q+\alpha,B_q):(e_r,D_r)\\(a_p,A):(b_q,B_q):(d_r,D_r)\end{matrix}\right)h(u)du \tag{19}$$

PROOF. It is evident from Lemma 2 that the first integral of (19) belong to $L_2(0,\infty)$ and thus we can apply the operator R to it. It is also true that the left hand side of (19) belongs to $L_2(0,\infty)$ and therefore the operator m can be applied to it by (7). Hence from (17) and (12) we obtain

$$m\left\{R\left(\frac{\alpha}{m_q},\ \frac{c_q}{B_q}:\frac{m_q}{B_q}\right)\int_0^\infty S_{p,q,r}(xu)h(u)du\right\}$$

$$=\frac{\sqrt{\left(\frac{c_q+sB_q}{m_q}\right)}}{\sqrt{\left(\frac{\alpha}{m_q}+\frac{c_q+sB_q}{m_q}\right)}}P(s)H(1-s)$$

$$=m\left\{\int_0^\infty S_{p,q,r}\left(xu\left|\begin{matrix}(c_{q-1},B_{q-1}),(c_q+\alpha,B_q):(e_r,D_r)\\(a_p,A):(b_q,B_q):(d_r,D_r)\end{matrix}\right.\right)h(u)du\right\} \tag{20}$$

The function in (20) operated upon by m, both belong to $L_2(0,\infty)$. Hence each side of (20) belong to $L_2\left(\frac{1}{2}-i\infty,\ \frac{1}{2}+i\infty\right)$ and so the operator m^{-1} can be applied to (20). Thus on applying m^{-1} to (20): the Lemma is established.

On proceeding in the same way, the following Lemma can be established easily.

LEMMA 4. *If* $x>0$, $h(x)\in L_2(0,\infty)$, $\beta>0$, $\left(\frac{e_j+1}{D_j}\right)>\frac{1}{2}$, $\frac{n_j}{D_j}>0$, $j=1,\cdots,r$ *then*

$$S\left(\frac{\beta}{n_r},\ \frac{e_r-D_r+1}{D_r},:\frac{n_r}{D_r}\right)\int_0^\infty S_{p,q,r}(xu)h(u)du$$

$$=\int_0^\infty S_{p,q,r}\left(xu\left|\begin{matrix}(c_q,B_q);(e_{r-1},D_{r-1});(e_r+\beta,D_r)\\(a_p,A);(b_q,B_q);(d_r,D_r)\end{matrix}\right.\right)h(u)du \tag{21}$$

6. Solution of the integral equation.

If (i) $d_j-e_j>0$: $(e_j-D_j+1)/D_j>\frac{1}{2}$, $n_j>0$, $j=1,\cdots,r$ (ii) $b_j-c_j>0$, $\frac{c_j}{B_j}>-\frac{1}{2}$, $m_j>0$, $j=1,\cdots,q$ (iii) $a_j-j>-1$, $\frac{j-1}{A}>-\frac{1}{2}$, $p>0$, $j=1,\cdots,p$ (iv) $x>0$ *then the solution of the integral equation*

$$\int_0^\infty S_{p,q,r}(xu)h(u)du=g(x) \tag{22}$$

is

$$h(x)=p^{-\frac{1}{2}}(2\pi)^{\frac{1}{2}-\frac{p}{2}}AL^{-1}\left[\prod_{j=1}^{p}R\left(\frac{a_j-(j-1)}{p},\ \frac{j-1}{A}:\frac{p}{A}\right)\right.$$
$$\left.\prod_{j=1}^{q}R\left(\frac{b_j-c_j}{m_j},\frac{c_j}{B_j}:\frac{m_j}{B_j}\right)\prod_{j=1}^{r}S\left(\frac{d_j-e_j}{n_j},\frac{e_j-D_j+1}{D_j}:\frac{n_j}{D_j}\right)g\left(\frac{Ax}{p}\right)\right] \tag{23}$$

where L^{-1} *denotes the inverse Laplace transform and* $h(x)\in L_2(0,\infty)$.

PROOF. It is evident that both sides of (22) belong to $L_2(0, \infty)$. Now applying the operator

$$S\Big(\frac{d_r-e_r}{n_r}, \frac{e_r-D_r+1}{D_r} : \frac{n_r}{D_r}\Big)$$

to (22), which is justified with the conditions stated, we obtain on using the result (21) that

$$\int_0^\infty S_{p,q,r-1}\Big(xu \Big| \begin{matrix}(c_q, B_q), (e_{r-1}, D_{r-1}) \\ (a_p, A), (b_q, B_q), (d_{r-1}, D_{r-1})\end{matrix}\Big) h(u)du$$

$$= S\Big(\frac{d_r-e_r}{n_r}, \frac{e_r-D_r+1}{D_r} : \frac{n_r}{D_r}\Big) g(x) \tag{24}$$

The above integral equation may be regarded as a reduction formula for the integral in (22).

Applying the operator S successively to (24) with d_r replaced by $d_{r-1}, d_{r-2}, \cdots, d_1$: e_r replaced by $e_{r-1}, e_{r-2}, \cdots, e_1$: n_r replaced by $n_{r-1}, n_{r-2}, \cdots, n_1$ and D_r replaced by $D_{r-1}, D_{r-2}, \cdots, D_1$ we get

$$\int_0^\infty S_{p,q,0}\Big(xu \Big| \begin{matrix}(c_q, B_q) \\ (a_p, A), (b_q, B_q)\end{matrix}\Big) h(u)du$$

$$= \prod_{j=1}^{r} S\Big(\frac{d_j-e_j}{n_j}, \frac{e_j-D_j+1}{D_j} : \frac{n_j}{D_j}\Big) g(x) \tag{25}$$

Now applying the operator,

$$R\Big(\frac{b_q-c_q}{m_q}, \frac{c_q}{B_q} : \frac{m_q}{B_q}\Big)$$

to (25), which is also justified by the conditions stated, we get on using the result (19) that

$$\int_0^\infty S_{p,q,-1,0}\Big(xu \Big| \begin{matrix}(c_{q-1}, B_{q-1}) \\ (a_p, A), (b_{q-1}, B_{q-1})\end{matrix}\Big) h(u)du$$

$$= R\Big(\frac{b_q-c_q}{m_q}, \frac{c_q}{B_q} : \frac{m_q}{B_q}\Big) \prod_{j=1}^{r} S\Big(\frac{d_j-e_j}{n_j}, \frac{e_j-D_j+1}{D_j} : \frac{n_j}{D_j}\Big) g(x) \tag{26}$$

The result (26) may also be regarded as a second reduction formula and thus applying the operator R successively to (26) with b_q replaced by $b_{q-1}, b_{q-2}, \cdots, b_1$: c_q and B_q replaced by $c_{q-1}, B_{q-1}, \cdots, c_1, B_1$: m_q replaced by $m_{q-1}, m_{q-2}, \cdots, m_1$ we get

$$\int_0^\infty S_{p,0,0}\Big(xu \Big| \overline{(a_p, A)}\Big) h(u)du$$

$$= \prod_{j=1}^{q} R\left(\frac{b_j-c_j}{m_j},\ \frac{c_j}{B_j} : \frac{m_j}{B_j}\right) \prod_{j=1}^{r} S\left(\frac{d_j-e_j}{n_j},\ \frac{e_j-D_j+1}{D_j} : \frac{n_j}{D_j}\right) g(x) \quad (27)$$

Now applying the operator

$$R\left(\frac{a_p-(p-1)}{p},\ \frac{p-1}{A} : \frac{p}{A}\right)$$

to equation (27) and using the result (19) we find that

$$\sqrt{\left(\frac{p-1+As}{p}\right.} \int_0^\infty S_{p-1,0,0}\left(xu \middle| \overline{(a_{p-1}, A)}\right) h(u)du$$

$$= R\left(\frac{a_p-(p-1)}{p},\ \frac{p-1}{A} : \frac{p}{A}\right) \prod_{j=1}^{q} R\left(\frac{b_j-c_j}{m_j},\ \frac{c_j}{c_j} : \frac{m_j}{c_j}\right)$$

$$\prod_{j=1}^{r} \left(\frac{d_j-e_j}{n_j}\ \frac{e_j-D_j+1}{D_j}\ \frac{n_j}{D_j}\right) g(x) \quad (28)$$

Applying this R operator successively to equation (28) with a_p replaced by a_{p-1}, $a_{p-2}, \cdots, a_1$; p replaced by $p-1, p-2, \cdots, 0$ and applying the Legendre's multiplication formula for the Gamma function [3, p. 5] we find that

$$(2\pi)^{\frac{1}{2}(p-1)} p^{\frac{1}{2}} A^{-1} \int_0^\infty \exp\left(-\frac{p_{xu}}{A}\right) h(u)du$$

$$= \prod_{j=1}^{p} R\left(\frac{a_j-(j-1)}{p},\ \frac{j-1}{A} : \frac{p}{A}\right) \prod_{j=1}^{q} R\left(\frac{b_j-c_j}{m_j},\ \frac{c_j}{B_j} : \frac{m_j}{B_j}\right)$$

$$\prod_{j=1}^{r} \left(\frac{d_j-e_j}{n_j},\ \frac{e_j-D_j+1}{D_j} : \frac{n_j}{D_j}\right) g(x) \quad (29)$$

which is equivalent to (23).

7. Particular cases.

If we set $A=B_1=\cdots=B_q=D_1=\cdots=D_r=1$, then the result (23) reduces to the inversion formula recently given by Saxena [11, p. 778].

(ii) For $p=m_i=n_j=1$; $i=1, \cdots, q$ and $j=1, \cdots, r$; $S_{p,q,r}(x)$ reduces to Fox's H-Function [5, p. 408] and the integral equation reduces to the following form:

If (i) $d_j-e_j>0, (e_j-D_j+1)/D_j>-\frac{1}{2}$; $j=1, \cdots, r$ (ii) $b_j-c_j>0, \frac{c_j}{B_j}>-\frac{1}{2}$; $j=1, \cdots, q$ (iii) $a_1>0$ (iv) $x>0$ (v) $h(x)$ is a solution of the integral equation.

$$\int_0^\infty H^{q+1,r}_{q+r,q+r+1}\left[xu \middle| \begin{matrix} (d_r, D_r), (c_q, B_q) \\ (a_1, A), (b_q, B_q), (e_r, D_r) \end{matrix}\right] h(u)du = g(x) \quad (30)$$

which belongs to $L_2(0,\infty)$ then

$$h(x)=(2\pi)^{\frac{1}{2}}AL^{-1}\Big\{R\Big(a_i,0:\frac{1}{A}\Big)\prod_{j=1}^{q}R\Big(b_j-c_j,\frac{c_j}{B_j}:\frac{1}{B_j}\Big)$$

$$\prod_{j=1}^{r}S\Big(d_j-e_j,\ \frac{e_j-D_j+1}{D_j}:\frac{1}{D_j}\Big)g(Ax)\Big\}$$

(iii) If we take $D_1=\cdots=D_r=A=B_1=\cdots=B_q=1$ in the result (30) then H-function reduces to Meijer's G-function [10] and on further puting $r=0$ we obtain an inversion formula given by Bhise [1].

It is interesting to mention here that the solution of various integral equations involving Bessel function [10], Whittaker function [13] etc can be derived from result (30) by giving special values to its parameters.

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