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EXPANSIONS OF MEIJER'S G-FUNCTION OF TWO VARIABLES WHEN THE UPPER PARAMETERS DIFFER BY INTEGERS

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O. O. MacRobert [6] obtained the series expansion of an E-function when two or three upper parameters differ by integers. The object of this article is to obtain expansions of Meijer's G-function of two variables when the upper parameters differ by integers. Series expansions are obtained in terms of Psi and generalized Zeta- functions and the G-function of two variables is represented in computable forms. The cases when two or three of the upper parameters differ by integers are given in detail and the technique is applicable when any number of parameters differ by integers. The representation of G-function of two variables in this article is suitable for computational purposes and thereby the results can be profitably used in practical problems. The technique developed in our article is different from that of MacRobert's and our technique can be applied to the situation when any number of parameters differ by integers.

1. Following Agarwal [1, p. 537] we define the G-function of two variables in terms of double Mellin-Barnes type integrals in the following form.

$$G^{n, v_1, v_2, m_1, m_2}_{p, (t_1, t_2), s, (q_1, q_2)}\left[\begin{array}{c|c} x & (\varepsilon_p), \\ & (\gamma_{t_1}):(\gamma'_{t_2}) \\ y & (\delta_s) \\ & (\beta_{v_1}):(\beta'_{v_2}) \end{array}\right]$$

$$(1.1)\quad = \frac{1}{(2\pi i)^2}\int_{L_2}\int_{L_1}\frac{\prod_{j=1}^{m_1}\Gamma(\beta_j+\xi)\prod_{j=1}^{v_1}\Gamma(\gamma_j-\xi)\prod_{j=1}^{m_2}\Gamma(\beta_j'+\eta)}{\prod_{j=m_1+1}^{q_1}\Gamma(1-\beta_j-\xi)\prod_{j=v_1+1}^{t_1}\Gamma(1-\gamma_j+\xi)\prod_{j=1+m_2}^{q_2}\Gamma(1-\beta_j'-\eta)}$$

$$\times\frac{\prod_{j=1}^{v_2}\Gamma(\gamma_j'-\eta)\prod_{j=1}^{n}\Gamma(1-\varepsilon_j-\xi-\eta)}{\prod_{j=n+1}^{p}\Gamma(\varepsilon_j+\xi+\eta)\prod_{j=1+v_2}^{t_2}\Gamma(1-\gamma_j'+\eta)\prod_{j=1}^{s}\Gamma(\delta_j-\xi-\eta)}x^{-\xi}y^{-\eta}d\xi d\eta$$

$$(1.2)\quad = \frac{1}{(2\pi i)^2}\int_{L_2}\int_{L_1}\prod_{j=1}^{2}\Gamma(\beta_j+\xi)\prod_{j=3}^{m_1}\Gamma(\beta_j+\xi)\prod_{j=1}^{m_2}\Gamma(\beta_j'+\eta)$$

$$\times\Delta_1(\xi)\Delta_2(\eta)\Delta_3(\xi+\eta)x^{-\xi}y^{-\eta}d\xi d\eta, \text{ say}$$

where

$$\Delta_1(\xi)=\frac{\prod_{j=1}^{m_1}\Gamma(\gamma_j-\xi)}{\prod_{j=m_1+1}^{q_1}\Gamma(1-\beta_j-\xi)\prod_{j=v_1+1}^{t_1}\Gamma(1-\gamma_j+\xi)},$$

$$\Delta_2(\eta)=\frac{\prod_{j=1}^{v_2}\Gamma(\gamma'_j-\eta)}{\prod_{j=1+m_2}^{q_2}\Gamma(1-\beta'_j-\eta)\prod_{j=1+v_2}^{t_2}\Gamma(1-\gamma'_j+\eta)}$$

$$\Delta_3(\xi+\eta)=\frac{\prod_{j=1}^{n}\Gamma(1-\varepsilon_j-\xi-\eta)}{\prod_{j=n+1}^{p}\Gamma(\varepsilon_j+\xi+\eta)\prod_{j=1}^{s}\Gamma(\delta_j-\xi-\eta)}$$

$0\le m_1\le q_1$, $0\le m_2\le q_2$, $1\le v_1\le t_1$, $1\le v_2\le t_2$, $0\le n\le p$

In what follows the symbol (a_p) will be used to represent the sequence of elements $a_1, a_2, \cdots, a_p$. The paths of integration are such that the poles of $\Gamma(\beta_j+\xi)$, $j=1, \cdots, m$, $\Gamma(\beta'_j+\eta)$ $j=1, 2, \cdots, m_2$ are separated from the poles of $\Gamma(\gamma_j-\xi)$, $j=1, \cdots, v_1$ $\Gamma(\gamma_{j'}-\eta)$ for $j=1, 2, \cdots, v_2$, $\Gamma(1-\varepsilon_j-\xi-\eta)$ for $j=1, 2, \cdots, n$. The integral converges, if

$$p+q_1+s+t_1<2(m_1+v_1+n)$$

$$p+q_2+s+t_2<2(m_2+v_2+n)$$

and

$$|\arg x|<\pi[m_1+v_1+n-\tfrac{1}{2}(p+q_1+s+t_1)]$$

$$|\arg y|<\pi[m_2+v_2+n-\tfrac{1}{2}(p+q_2+s+t_2)]$$

In case $p+t_1=s+q_1$ and $p+t_2=s+q_2$

we must also have

$$|x|+|y|<\min(1, 2^{s-p+1}).$$

By using Braaksma's theorem [2, p. 278, theorem 6. 1] we can evaluate the G-function of (1.1) as the sum of the residues at the poles of $\Gamma(\beta_j+\xi)$ for $j=1, 2, \cdots, m_1$, $\Gamma(\beta'_j+\eta)$ for $j=1,2,\cdots, m_2$.

For simplicity we will consider two cases but it is easy to see that the method developed can be used for other cases as well.

Case I. Let $\beta_2=\beta_1+n'$, $n'=0, 1, \cdots\cdots$. Let none of the other β_j $j=3, 4,$

$\cdots$, m_1 differ by integers and none of the poles of $\Gamma(\beta_j+\xi)$ $j=1$, 2 coincides with any zeros of the denominator Gammas. In this case a series expansion of (1.1) will be obtained with the help of calculus of residues. It can be easily seen that (1.1) equals,

$$(1.3)\quad \sum_{v=0}^{n'-1}\sum_{j'=1}^{m_2}\sum_{n_j'=0}^{\infty}\frac{(-1)^{v+n_{j'}}\prod_{j=2}^{m_1}\Gamma(\beta_j-\beta_1-v)\prod_{j=1}^{m_2}\Gamma(\beta'_j-\beta'_{j'}-n_{j'})}{v!n_{j'}!}$$

$$\times\Delta_1(-\beta_1-v)\Delta_2(-\beta'_{j'}-n_{j'})\Delta_3(-\beta_1-v-\beta'_{j'}-n_{j'})x^{\beta_1+v}y^{\beta'_{j'}+n_{j'}}$$

$$+\sum_{j'=1}^{m_2}\sum_{n_{j'}=0}^{\infty}\sum_{j=3}^{m_1}\sum_{n_j=0}^{\infty}\frac{(-1)^{n_j+n_{j'}}\prod_{h=1}^{m_1}{}'\Gamma(\beta_h-\beta_j-n_j)\prod_{h=1}^{m_2}{}'\Gamma(\beta'_h-\beta'_{j'}-n_{j'})}{n_j!n_{j'}!}$$

$$\times\Delta_1(-\beta_1-n_j)\Delta_2(-\beta'_{j'}-n_{j'})\Delta_3(-\beta_j-n_j-\beta'_{j'}-n_{j'})$$

$$\times x^{\beta_j+n_j}y^{\beta'_{j'}+n_{j'}}+\sum_{v=n'}^{\infty}\sum_{j'=1}^{m_2}\sum_{n_j'=0}^{\infty}\frac{(-1)^{n'+n_{j'}}\prod_{j=3}^{m_1}\Gamma(\beta_j-\beta_1-v)}{v!(v-n)!\ n_{j'}!}$$

$$\times\prod_{h=1}^{m_2}{}'(\beta'_h-\beta'_{j'}-n_{j'})\Delta_1(-\beta_1-v)\Delta_2(-\beta'_{j'}-n_{j'})\Delta_3(-\beta_1-v-\beta'_{j'}-n_{j'})$$

$$\times x^{\beta_1+v}y^{\beta'_{j'}+n_{j'}}[-\log x+\sum_{j=3}^{m_1}\phi(\beta_j-\beta_1-v)+\phi_{11}(-\beta_1-v)$$

$$+\phi_{31}(-\beta_1-v-\beta'_{j'}-n_{j'})+\phi(v+1)+\phi(v-n+1)],$$

where

$$(1.4)\quad \phi_{11}(-\beta_1-v)=\frac{\partial}{\partial\xi}\log\ \Delta_1(\xi)\ \text{at}\ \xi=-\beta_1-v$$

$$(1.5)\quad \phi_{31}(-\beta_1-v-\beta_{j'}-n_{j'})=\frac{\partial}{\partial\xi}\log\ \Delta_3(\xi+\eta)\ \text{at}\ \xi=-\beta_1-v$$

and $\eta=-\beta'_{j'}-n_{j'}$ and for example,

$$(1.6)\quad \phi_{11}(-\beta_1-v)=\sum_{j=1}^{v_1}\phi(\gamma_j+\beta_1+v)-\sum_{j=m_1+1}^{v_1}\phi(1-\beta_j+\beta_1+v)-\sum_{j=v_1+1}^{t_1}\phi(1-\gamma_j-\beta_1-v)$$

and the ϕ-function is defined as,

$$(1.7)\ \phi(z)=(d/dz)\ \log\ \Gamma(z)=-\gamma+(z-1)\sum_{n=0}^{\infty}\frac{1}{(n+1)(z+n)}$$

where γ is the Euler's constant; $\gamma=-\phi(1)=0.5772156649\cdots$.

A Particular case.

$$G^{0,\ 0,\ 0,\ 2,\ 2}_{2,\ (0,\ 2),\ 0,\ (2,\ 2)}\left[\begin{array}{c|c} & \varepsilon_1,\varepsilon_2 \\ x & -;\ \gamma_1',\gamma_2' \\ y & -\cdots- \\ & \beta_1,\beta_2;\beta_1',\beta_2' \end{array}\right]$$

$$(1.8) \quad = \frac{1}{(2\pi i)^2} \int_{L_2} \int_{L_1} \frac{\Gamma(\beta_1+\xi)\Gamma(\beta_2+\xi)\Gamma(\beta_1'+\eta)\Gamma(\beta_2'+\eta)x^{-\xi}y^{-\eta}d\xi d\eta}{\Gamma(1-\gamma_1'+\eta)\Gamma(1-\gamma_2'+\eta)\Gamma(\varepsilon_1+\xi+\eta)\Gamma(\varepsilon_2+\xi+\eta)}$$

Let $\beta_2=\beta_1+n$, $n=0, 1, \cdots$; $\beta_1-\beta_2\neq\pm\lambda$, $\lambda=0, 1, 2, \cdots$, then (1.8) equals

$$(1.9) \quad \sum_{v=0}^{n-1}\sum_{j=1}^{2}\sum_{v_j=0}^{\infty} \frac{(-1)^{v+v_j}\Gamma(n-v)\prod_{h=1}^{2}{}'\Gamma(\beta'_h-\beta'_j-v_j)x^{\beta_1+v}y^{\beta_j'+v_j}}{v!\ v_j!\Gamma(1-\gamma_1'-\beta'_j-v_j)\ \Gamma(1-r_2'-\beta_j'-v_j)\Gamma(\varepsilon_1-v-\beta_j'-v_j)}$$

$$\times \frac{1}{\Gamma(\varepsilon_2-v-\beta'_j-v_j)} + \sum_{v=n}^{\infty}\sum_{j=1}^{2}\sum_{v_j=0}^{\infty} \frac{(-1)^{n+v_j}\Pi'\Gamma(\beta_h'-\beta_j'-v_j)}{v!\ (v-n)!\ v_j!\ \Gamma(\varepsilon_1-v-\beta'_j-v_j)}$$

$$\times \frac{x^{\beta_1+v}y^{\beta'_j+v_j}}{\Gamma(1-\gamma_1'-\beta_j'-v_j)\Gamma(1-r_2'-\beta_j'-v_j)\Gamma(\varepsilon_2-v-\beta'_j-\gamma_j)}[-\log x+\phi(v+1)$$

$$+\phi(v-n+1)-\phi(\varepsilon_1-v-\beta'_j-v_j)-\phi(\varepsilon_2-v-\beta'_j-v_j)]$$

From (1.8) we can get an interesting result as follows:

$$(1.10) \quad \lim_{\substack{y\to 0 \\ \beta_1'=0,\beta_2'>0}} G^{0,\ 0,\ 0,\ 2,}_{2,\ (0,\ 2),\ 0,\ (2,\ 2)} \left[\begin{array}{c|c} & \varepsilon_1, \varepsilon_2 \\ x & -;\gamma_1', \gamma_2' \\ y & - \\ & \beta_1, \beta_2; \beta_1', \beta_2' \end{array} \right]$$

$$= \frac{\Gamma(\beta_2')}{\Gamma(1-\gamma'_1)\Gamma(1-\gamma'_2)} G^{2,\ 0}_{2,\ 2}\left(x \,\middle|\, \begin{array}{l} \varepsilon_1,\ \varepsilon_2 \\ \beta_1,\ \beta_2 \end{array}\right)$$

$$= \frac{\Gamma(\beta_2')}{\Gamma(1-\gamma_1')\Gamma(1-\gamma_2')} \left\{ \sum_{v=0}^{n-1} \frac{(-1)^v\Gamma(n-v)!\ x^{\beta_1+v}}{v!\ \Gamma(\varepsilon_1-v)\Gamma(\varepsilon_2-v)} \right.$$

$$+ \sum_{v=n}^{\infty} \frac{(-1)^n x^{\beta_1+v}[-\log x+\phi(v+1)+\phi(v-n+1)-\phi(\varepsilon_1-v)-\phi(\varepsilon_2-v)]}{v!(v-n)!\ \Gamma(\varepsilon_1-v)\ \Gamma(\varepsilon_2-v)}$$

By using the integral representation given in [5] we can easily see that if $\varepsilon_1=a+1/3$, $\varepsilon_2=a+2/3$, $\beta_1=a$, $\beta_2=a$, then

$$(1.12) \quad G^{2,\ 0}_{2,\ 2}\left[x \,\middle|\, \begin{array}{cc} a+1/3, & a+2/3 \\ a, & a \end{array}\right] = x^a{}_2F_1(2/3,\ 1/3\ ;\ 1\ ;\ 1-x)$$

Case II. Let $\beta_2=\beta_1+n'$, $\beta_3=\beta_2+m'=\beta_1+n'+m'$. None of the other β's differ by integers. None of the poles of $\prod_{j=1}^{3}\Gamma(\beta_j+\xi)$ coincide with any of the zeros of the denominator Gammas. In this case (1.1) can have poles of order 1, 2; and 3. The residue corresponding to these poles will be evaluated by using the following techniques. If Δ denotes a Gamma product having a pole of order k at $\xi=-a$ then the residue of $\Delta x^{-\xi}$ at $\xi=-a$ is given by

$$(1.13)\quad \frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial\xi^{k-1}}(\xi+a)^k\Delta x^{-\xi} \text{ at } \xi=-a$$

$$=\frac{x^{-\xi}}{(k-1)!}\left\{\frac{\partial}{\partial\xi}+(-\log x)\right\}^{k-1}(\xi+a)^k\Delta, \text{ at } \xi=-a$$

$$(1.14)\quad =\frac{x^{-\xi}}{(k-1)!}\sum_{v=0}^{k-1}\binom{k-1}{v}(-\log x)^{k-v-1}\frac{\partial^v}{\partial\xi^v}(\xi+a)^k\Delta, \text{ at } \xi=-a$$

But

$$(1.15)\quad \frac{\partial^v}{\partial\xi^v}(\xi+a)^k\Delta=\frac{\partial^{v-1}}{\partial\xi^{v-1}}A\Delta, \text{ where}$$

$$A=\frac{\partial}{\partial\xi}\log\,(\xi+a)^k\Delta$$

It will be more convenient to use (1.15) when Δ is a Gamma product. By using the above technique (1.1) under case II can be easily worked out and the result can be easily seen to be as follows.

$$(1.16)\quad \sum_{v=0}^{n'-1}\sum_{j'=1}^{m_2}\sum_{n_j'=0}^{\infty}\frac{(-1)^{v+n_{j'}}\prod_{j=2}^{m_1}\Gamma(\beta_j-\beta_1-v)\prod_{j=1}^{m_2}{}'\Gamma(\beta_j'-\beta'_{j'}-n_j')}{v!n_j'!}$$

$$\times\Delta_1(-\beta_1-v)\Delta_2(-\beta'_{j'}-n_j')\Delta_3(-\beta_1-v-\beta'_{j'}-n_{j'})x^{\beta_1+v}y^{\beta'_{j'}+n_{j'}}$$

$$+\sum_{j'=1}^{m_2}\sum_{n_j'=0}^{\infty}\sum_{v=n'}^{n'+m'-1}\frac{(-1)^{n'+n_{j'}}\prod_{j=3}^{m_1}\Gamma(\beta_j-\beta_1-v)\prod_{j=1}^{m_2}\Gamma(\beta_{j'}-\beta'_{j'}-n_j')}{n'!(v-n')!\;n_j'!}$$

$$\times\Delta_1(-\beta_1-v)\Delta_2(-\beta'_{j'}-n_j')\Delta_3(-\beta_1-v-\beta'_{j'}-n_{j'})x^{\beta_1+v}y^{\beta'_{j'}+n_{j'}}$$

$$\times[-\log x+\sum_{j=3}^{m_1}\phi(\beta_j-\beta_1-v)+\phi_{11}(-\beta_1-v)+\phi_{31}(-\beta_1-v-\beta'_{j'}-n_{j'})$$

$$+\phi(v+1)+\phi(v-n+1)]+\sum_{j'=1}^{m_2}\sum_{n_{j'}=0}^{\infty}\sum_{v=n'+m'}^{\infty}\frac{(-1)^{3v-2n'-m'+n_{j'}}}{v!(v-n')!}$$

$$\times\frac{\prod_{j=4}^{m_1}\Gamma(\beta_j-\beta_1-v)\prod_{j=1}^{m_2}\Gamma(\beta_{j'}-\beta'_{j'}-n_{j'})}{(v-n'-m')!\;n_{j'}!}\Delta_1(-\beta_1-v)\Delta_2(-\beta'_{j'}-n_{j'})$$

$$\times\Delta_3(-\beta_1-v-\beta'_{j'}-n_{j'})x^{\beta_1+v}y^{\beta'_{j'}+n_{j'}}\frac{1}{2!}[A^1+A^2+2(-\log x)+(-\log x)^2]$$

$$+\sum_{j'=1}^{m_2}\sum_{n_j'=0}^{\infty}\sum_{j=4}^{m_1}\sum_{n_j=0}^{\infty}\frac{(-1)^{n_j+n_{j'}}\prod_{h=1}^{m_1}{}'\Gamma(\beta_h-\beta_j-n_j)\prod_{h=1}^{n_2}{}'\Gamma(\beta'_h-\beta'_{j'}-n_{j'})}{n_j!\ n_j'!}$$

$$\times[\Delta_1(-\beta_j-n_j)\Delta_2(-\beta'_{j'}-n_{j'})\Delta_3(-\beta_j-n_j-\beta'_{j'}-n_{j'})x^{\beta_j+n_j}y^{\beta'_{j'}+n_{j'}}]$$

where

$$A=\phi(v+1)+\phi(v-n'+1)+\phi(v-n'-m'+1)+\sum_{j=4}^{m'}\phi(\beta_j-\beta_1-v)$$

$$+\phi_{11}(-\beta_1-v)+\phi_{31}(-\beta_1-v-\beta'_{j'}-n_j')$$

$$A'=\zeta(2,v+1)+\zeta(2,v-n'+1)+\zeta(2,\ v-n'-m'+1)+\sum_{j=4}^{m'}\zeta(2,\beta_j-\beta_1-v)$$

$$+\zeta_{11}(2,-\beta_1-v)+\zeta_{31}(2,-\beta_1-v-\beta'_{j'}-n_{j'})$$

where $\phi_{11}(\ .\ )$ and $\phi_{31}(\ .\ )$ are given in (1.4) and (1.5) and for example,

$$(1.19)\qquad \zeta_{11}(2,-\beta_1-v)=\frac{\partial}{\partial\xi}\phi_{11}(\xi),\ \text{at}\ \xi=-\beta_1-v$$

$$(1.20)\qquad \zeta_{31}(2,\ -\beta_1-v-\beta'_{j'}-n_{j'})=\frac{\partial}{\partial\xi}\phi_{31}(\xi)\ \text{at}\ \xi=-\beta_1-v\ \text{and}\ \eta=-\beta'_{j'}-n_{j'}$$

the generalized Riemann Zeta function $\zeta\ (.\ ,\ .\ )$ is defined as,

$$(1.21)\qquad \zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s},\ a\neq 0,\ -1,\ -2,\ \cdots,\ R(s)>1$$

REMARKS.

(i) The technique given in this article can be extended to the cases where a number of β_j, $j=1,2,\cdots,m_1$ differ by integers or β_j, $j=1,2,\cdots,m_1$ and β'_j, $j=1,2,\cdots,m_2$ differ by integers.

(ii) From the structure of (1.3) and (1.16) it is easily seen that several results on G-functions of two variables can be extended to cover the cases when the upper parameters differ by integers. For the purpose of illustration we will represent in computable form the following integral, which easily follows from [4, p. 399]

$$m^{\gamma}\int_0^1 u^{\gamma-1}(1-u)^{\rho-1}{}_2F_1(\alpha,\ \beta;\ \gamma;\ u)$$

$$\times G^{n,\ v_1,\ v_2,\ m_1,\ m_2}_{p,(t_1,\ t_2),\ s,\ (v_1,\ v_2)}\begin{bmatrix}x(1-u)^m\\ y(1-u)^m\end{bmatrix}du$$

$$(1.22)\qquad = G^{n+2m,\ v_1,\ v_2,\ m_1,\ m_2}_{p,\ (t_1,\ t_2),\ s+2m,\ (q_1,\ q_2)}\left[\begin{array}{c|c} x & \Delta(m,\ 1\text{-}\rho),\ \Delta(m,\ \gamma-\alpha-\beta+\rho),\ (\varepsilon_p) \\ & (\gamma_{t_1});\ (\gamma'_{t_2}) \\ y & (\delta s),\Delta(m,\ \gamma-\alpha+\rho),\ \Delta(m,\ \gamma-\beta+\rho) \\ & (\beta_{v_1});\ (\beta'_{v_2}) \end{array}\right]$$

where $R(\gamma)$ 0, $R(\rho+m\beta_{j_1}+m\beta'_{j_2})>0$, $R(\rho+\gamma-\alpha-\beta+m\beta_{j_1}+m\beta_{j_2})>0$ for $j_1=1$, 2, $\cdots$, m_1, ; $j_2=1$, 2, $\cdots$, m_2 and other conditions are as listed in (1.1). Here for example $\Delta(r,\ a)$ denotes the sequence of parameters a/r, $(a+1)/r$, $\cdots$, $(a+r-1)/r$.

Now the expansions of (1.22) under cases I and II are given in (1.3) and (1.16) respectively with the following replacements: $n\sim n+2m$, $s\sim s+2m$, $(\varepsilon_p)\sim[\Delta(m,\ 1-\rho),\ \Delta(m,\ \gamma-\alpha-\beta+\rho),\ (\varepsilon_p)]$; $(\delta_s)\sim[(\delta_s),\ \Delta(m,\ \gamma-\alpha+\rho),\ \Delta(m,\ \gamma-\beta+\rho)]$ where $\sim$means 'replaced by'.

It is easy to notice that several such results, when the upper parameters $-s$ differ by integers can be put into computable forms by using the techniques given in this article.

(iii) Series expansions of Meijer's G-function [3, p.207], under the cases I and II can be worked out by using the identity [1, p. 539(iv)].

MacRobert's results [6, 7] can also be derived from (1.3) and (1.16) on employing the formula

$$\lim_{y\to 0} G^{1,\ 0,\ 0,\ P,\ 1}_{Q+1,\ (0,\ 0),\ 0,\ (P,\ 1)}\left[\begin{array}{c|c} & 1,\ (b_Q) \\ x & \cdots\cdots\cdots\cdots \\ y & \cdots\cdots\cdots\cdots \\ & (a_P),\ (0) \end{array}\right]$$

$$= G^{P,\ 1}_{Q+1,\ P}\left[x\,\middle|\,\begin{matrix}1,\ b_1,\ \cdots,\ b_Q \\ a_1,\ \cdots,\ a_P\end{matrix}\right] = E(P;a_r:Q;\ b_s:x)$$

where E denotes the MacRobert's E-function [3, p.205].

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